Scratching the Surface of Topological Surfaces

#74 of Gottschalk's Gestalts

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GG74-2

C. a brief account of compact topological surfaces

☐ a closed topological surface S
$=_{df}$ a compact connected locally planar Hausdorff space;
a compact topological surface
$=_{df}$ a closed topological surface from which
a finite number r of open disks
with pairwise disjoint closures
have been removed;
the fundamental theorem
on compact topological surfaces
states that
a compact topological surface S
is uniquely topologically characterized by
• the orientability of S
&
• either the genus g of S
or the Euler characteristic χ of S
&
• the cuff number r of S

☐ the orientability of

a compact topological surface S

$=_{df}$ whether S is orientable ($=_{ab}$ or) = two - sided

wiet

some triangulation of S is orientable

wiet

all triangulations of S are orientable

or

whether S is nonorientable ($=_{ab}$ nor) = one - sided

wiet

some triangulation of S is not orientable

wiet

all triangulations of S are not orientable

□ for an orientable
compact topological surface S,
S $=_{top}$ the sphere with p handles & r cuffs
wh p, r $\in$ unique nonneg ints
&
its genus g = p

□ for a nonorientable
compact topological surface S,
S $=_{top}$ the sphere with q crosscaps & r cuffs
wh q $\in$ unique pos int & r $\in$ unique nonneg int
&
its genus g = q

□ a handle is a torus with one hole
&
a crosscap is a Möbius band;
either may be glued to
a hole (= disk removed) in the sphere
by identifying boundaries
wa simple closed curves

☐ the Euler characteristic
of any compact topological surface S
$= \chi = V - E + F$
wh
$V = \#$ vertices
$E = \#$ edges
$F = \#$ faces
in any triangulation of S

☐ $S(p, q, r)$ = the sphere
with p handles & q crosscaps & r cuffs
wh $p, q, r \in$ nonneg int
has Euler characteristic
$\chi = 2 - 2p - q - r$

☐ the genus g of a closed topological surface S
$=_{df}$ the maximum number
of pairwise disjoint simple closed curves in S
along which S can be cut
without disconnecting S;
the genus g of a compact topological surface
$=_{df}$ the genus of the corresponding
closed topological surface,
the cuff number not affecting the genus

☐ S(p, q, r) = the sphere
with p handles & q crosscaps & r cuffs
wh p, q, r $\in$ nonneg int
has genus

$g = p = \frac{1}{2}(2-\chi-r)$ if $S \in$ or wiet $q = 0$

&

$g = 2p+q = 2-\chi-r$ if $S \in$ nor wiet $q \neq 0$

E. some closed topological surfaces

• the sphere

or

$g = 0$

$\chi = 2$

$p = 0$

$q = 0$

$r = 0$

• the torus

or

$g = 1$

$\chi = 0$

$p = 1$

$q = 0$

$r = 0$

• the pretzel with 2 arches

= the 2 - arch pretzel

= the 2 - pretzel

= the double pretzel

= the pretzel

or

$g = 2$

$\chi = -2$

$p = 2$

$q = 0$

$r = 0$

• the pretzel with 3 arches

= the 3 - arch pretzel

= the 3 - pretzel

= the triple pretzel

or

$g = 3$

$\chi = -4$

$p = 3$

$q = 0$

$r = 0$

• the pretzel with n arches

= the n - arch pretzel

= the n - pretzel

wh $n \in$ int ≥ 2

or

$g = n$

$\chi = 2 - 2n$

$p = n$

$q = 0$

$r = 0$

• the projective plane

nor

$g = 1$

$\chi = 1$

$p = 0$

$q = 1$

$r = 0$

• the Klein bottle

nor

$g = 2$

$\chi = 0$

$p = 0$

$q = 2$

$r = 0$

E. some compact topological surfaces

• the disk

or

$g = 0$

$\chi = 1$

$p = 0$

$q = 0$

$r = 1$

• the cylinder

or

$g = 0$

$\chi = 0$

$p = 0$

$q = 0$

$r = 2$

• the T - joint

or

$g = 0$

$\chi = -1$

$p = 0$

$q = 0$

$r = 3$

• the X - joint

= the T - shirt

or

$g = 0$

$\chi = -2$

$p = 0$

$q = 0$

$r = 4$

• the n - joint wh $n \in \text{int} \geq 2$

or

$g = 0$

$\chi = 2 - n$

$p = 0$

$q = 0$

$r = n$

- the handle

or

$g = 1$

$\chi = -1$

$p = 1$

$q = 0$

$r = 1$

- the carryall

or

$g = 1$

$\chi = -2$

$p = 1$

$q = 0$

$r = 2$

- the amphora

or

$g = 2$

$\chi = -3$

$p = 2$

$q = 0$

$r = 1$

• the Möbius band / strip = the crosscap

nor

$g = 1$

$\chi = 0$

$p = 0$

$q = 1$

$r = 1$

C. other topological surfaces

□ an open topological surface
$=_{df}$ a noncompact connected locally planar
second - countable Hausdorff space;
second - countable
(= the existence of a countable base of open sets)
is imposed
to make the surface (equivalently) triangulable;
the classification of open topological surfaces
is not so simple;
the terminology of 'closed' and 'open'
as applied to topological surfaces
goes back to over a century ago when
the use of these words
had not yet been standardized
to sets of topological spaces