Math Snatches & Patches

#58 of Gottschalk's Gestalts

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of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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D. asymptotic equality

let

- $D \subset \mathbb{R}$
- D contains a sequence going to $\infty = +\infty$
- $f, g\colon D \to \mathbb{R}$ st $g(x) \neq 0$ for $x \in D$
- $x \in$ var D

then

- f(x) is asymptotically equal to g(x) at infinity

= f(x) asymptotically equals g(x) at infinity

= f(x) and g(x) are asymptotically equal at infinity

= f(x) is asymptotic to g(x) as x goes to infinity

= f(x) and g(x) are asymptotic as x goes to infinity

= f(x) is asymptotic to g(x) as $x \to \infty$

= f(x) and g(x) are asymptotic as $x \to \infty$

$=_{dn}$ $f(x) \sim g(x)$ as $x \to \infty$

$=_{df}$ $\exists \lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$

D. Landau's big – oh & little - oh notation

let

- $D \subset \mathbb{R}$
- D contains a sequence going to $\infty = +\infty$
- f, g: $D \to \mathbb{R}$ st $g(x) \neq 0$ for $x \in D$
- $x \in$ var D

then

- f(x) equals / is big – oh of g(x)

as x goes to infinity

$=_{dn}$ $f(x) = O(g(x))$ as $x \to \infty$

$=_{df}$ $\left|\dfrac{f(x)}{g(x)}\right| \in$ bounded for x sufficently large

- f(x) equals / is little – oh of g(x)

as x goes to infinity

$=_{dn}$ $f(x) = o(g(x))$ as $x \to \infty$

$=_{df}$ $\dfrac{f(x)}{g(x)} \to 0$ as $x \to \infty$

note: some writers have 'zero' in place of 'oh'

□ the Bell numbers

• the nth Bell number ($n \in$ nonneg int)
$=_{dn}$ B(n) wh B ← Bell
$=_{df}$ the number of partitions
of a set with cardinal n

• the Bell numbers B(n) ($0 \leq n \leq 10$)
B(0) = 1
B(1) = 1
B(2) = 2
B(3) = 5
B(4) = 15
B(5) = 52
B(6) = 203
B(7) = 877
B(8) = 4140
B(9) = 21147
B(10) = 115975

• how to construct

the triangle generating Bell numbers

= the Bell triangle

there are four rules:

(1) rows begin at the left - hand margin

(2) the end of one row is the beginning of the next row

(3) once another row is started,

it is continued by repetitive use of the pattern

a

+

b $\to$ c

until its application

is no longer possible

(4) the first row consists of the single entry 1

• the Bell triangle

1						
1	2					
2	3	5				
5	7	10	15			
15	20	27	37	52		
52	67	87	114	151	203	
203	255	322	409	523	674	877

…

• a generating function for the Bell numbers

$$\exp \exp x = e^{e^x}$$

$$= e \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}$$

$$= e\left(1 + \frac{1x}{1!} + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \cdots\right)$$

• also note

$$\sum_{n=0}^{\infty} \frac{n^k}{n!} = B(k)\, e \quad (k \in \text{nonneg int})$$

• bioline

Eric Temple Bell

1883-1960

Scottish-American

number theorist, historian of mathematics;

wrote the books

Men of Mathematics (1937)

The Development of Mathematics (1940)

Mathematics, Queen and Servant of Science (1951)

The Last Problem (1961);

under the pen name of John Taine

he wrote many books of science fiction including

The Time Stream (1946)

☐ sizes of sets

let A be a set
then in each item tsape

• A is the zero-element set
A is the empty set
A is empty
$A = \varnothing$
crd A = 0

• A is a one-element set
A is a single element set
A is a singleton
crd A = 1

• A is a two-element set
A is a double element set
A is a doubleton
crd A = 2

• A is a three-element set
A is a triple element set
A is a tripleton
crd A = 3

• A is an n-element set ($n \in$ nonneg int)
crd A = n

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• A is a nonempty set

A is nonempty

$A \neq \varnothing$

crd $A \geq 1$

• A is a many element set

A is a multiple element set

A is a pluralton

A is plural

crd $A \geq 2$

• A has at least n elements ($n \in$ nonneg int)

crd $A \geq n$

• A has at most n elements ($n \in$ nonneg int)

crd $A \leq n$

• A has exactly/precisely n elements ($n \in$ nonneg int)

crd $A = n$

• A is a finite set
A is finite
A has only finitely many elements
crd A $< \aleph_0$

• A is an infinite set
A is infinite
A has infinitely many elements
crd A $\geq \aleph_0$

• A is a countable set
A is countable
A has only countably many elements
crd A $\leq \aleph_0$

• A is an uncountable set
A is uncountable
A has uncountably many elements
crd A $> \aleph_0$
crd A $\geq \aleph_1$

• A is a countably infinite set
A is a denumerable set
A is countably infinite
A is denumerable
crd A $= \aleph_0$

• A has the cardinal/power of the continuum
$\mathrm{crd}\,A = \mathrm{crd}\,\mathbb{R} = 2^{\aleph_0} = \mathbb{c}$

☐ circled letters

• the circled sign @
comes from
business/commerce/finance
where it means (and may be read)
at
each
at the price of ... each
and the like,
these paraphrases all meaning
pretty much the same thing;
eg
5 apples @$2 = $10
means
five apples at two dollars each
costs ten dollars altogether

• in recent times the circled sign @
is used in an e-mail address as
recipient@location.domain
and is read 'at' in that context;
it was first used for this purpose in 1971 by
the inventor of e-mail Ray Tomlinson
in the first e-mail message;
in his words:
'The @ sign seemed to make sense.
I used the @ sign to indicate that the user
was 'at' some other host rather than being local.'

• the mark @
consists of a lower case letter a
with a circumscribing circle (almost)
and may be called
the circled a
where a in this context comes from
the initial letter of 'at';
the mark @
may be made cursively in a lovely manner
and this manner is reproduced in this typefont, I note;
the symbol @ and this usage is likely centuries old;
I could not immediately find out
its first printed occurrence
but I would guess it occurs first
in some textbook for business arithmetic,
certainly no later than the early19th century, I should think

• other circled letters occur in general use;
eg
circled c
∀
means 'copyright'
&
circled R
∀
means 'registered'

• some circled math signs occur
from time to time;
eg
circled plus +
$\oplus$
read 'oh plus'
&
circled times $\times$
$\otimes$
read 'oh times'
may be used to denote certain binary operations
which are called addition & multiplication

☐ names for the sign ∞

- lazy eight = the figure eight lying on its side
- love knot = a looped cord tying a couple together
  by enclosing each
- twisted ouroboros = a twisted serpent
  biting/swallowing its tail
- lemniscate
- infinity
- infinity sign

note: in 1655
the English mathematician John Wallis
used the sign ∞ to mean infinity;
∞ is an ancient Roman sign for 1000;
its choice was possibly influenced by
its resemblance to little omega ω,
the lowercase form of the last letter
of the Greek alphabet

☐ a word contributing to the origin of the word mathematics

- μαϑημα (Greek)

= μαϑησις (Greek)

= mathesis (Latin)

= all knowledge

□ to subitize

= pr tuh SOO-bit-tize

= ri subito (Latin) = suddenly, immediately

= df to perceive at a glance
the number of items present
without actually counting or grouping them;
ordinarily this is possible for up to say five items

where
the three equaters
have the meanings:
= pr means is pronounced
= ri means is rooted in
= df means is defined to mean

☐ a fanciful derivation of the word
radian
in successive abbreviations

• the measure of
the central angle of a circle
subtended by an arc
whose length is
the radius of the circle

= angle subtended by radius

= radial angle

= radi an

= radian

• the word radian
was introduced in 1873
by James Thomson,
the brother of Lord Kelvin;
I do not know his thoughts
at the time

□ synonymous words & phrases

therefore
hence
consequently
thus
whence
ergo (Latin)

as a conclusion
as a consequence
as a result
it is a conclusion that
it is a consequence that
it is a result that
it follows that
there follows
we conclude that
we draw the conclusion that

etc

□ understanding & communicating mathematics

• i do not believe that
the understanding
of a given piece of mathematics by a given person
can be fully described & accurately measured
by a simple percentage from 0% to 100% ;
by 'understanding mathematics' i do not mean
memorizing theorems, proofs, and examples,
nor solving designed problems,
nor performing simple algorithms;
i have in mind something that likely includes those things
but is much more;
i believe that i can more or less recognize it
but i am hard put to it to describe it adequately;
research in the topic would be evidence
of the 'understanding' of the topic;
on the basis of my seven and a fraction decades of
learning/teaching/reading/hearing/talking/researching
mathematics i am inclined to think that
the understanding of mathematics
is peculiar to each individual;
as to how someone understands mathematics
varies from person to person;
even among experts it is variable;
not just in quantity
but in quality also

• different people
must understand mathematics
in different ways;
but how to classify these ways
& how to recognize
a particular way in a particular person
remain mysteries

• ¿ is there just one 'correct' understanding of mathematics ?
i doubt it;
but to be sure 'correct understandings'
should be sought
among the experts & not the tyros;
¿ can 'correct understandings' be linerarly ordered
in measure of merit ?

• of course
there is the question
¿ is there an essential distinction between
mathematics
&
the understanding of mathematcs ?
i am inclined to believe
there is indeed an essential distinction;
consider that
'understanding mathematics'
is attached to one person
but
'mathematics'
is not

- i also recognize that

a person may have a clear understanding of mathematics
&
yet be unable to clearly communicate that understanding;
i guess all students & teachers of math
know this in very many minor ways;
but it is also true in major ways;
i think that i can recognize
highly nontrivial examples in the history of mathematics
where a mathematician saw clearly
something of great importance
but the language/theory had not yet been developed
for an exact expression of that piece of mathematics;
this is evidence that
mathematics is helped by language
but
mathematics is not to be identified with language;
to be sure mathematicians are
tremendously dependent upon both
natural language
&
especially devised mathematical symbolism;
without either
mathematicians would get just about nowhere

• as a practical matter
a natural language such as English
is necessary for mathematics
in recorded expression & in communication
if for no less

• ¿ is there a natural language
that is especially appropriate
for mathematics ?

• ¿ for native speakers
is it just as easy
to think & understand math
in German as it is in English say ?

• i have been told that a person
always does arithmetic in their native language;
it is much easier that way
likely because
the simple algorithms are learned at a young age
in the native language

• i have had the experience of making some remark
in the course of an explanation
to an undergraduate student
& had it seized upon by the student
as the magic incantation
for understanding some particular point;
they exclaimed
'So that's what it means !'
or even
'Why didn't you say so before?';
my own attitude to the remark was likely to be that
the remark was painfully evident
& not especially insightful anyway;
yet many students must feel that
math teachers keep all that math secret
by not saying just the right word
for them to understand it all

• again
i have had the experience of a student
with a high grade & who could 'work the problems'
saying a good deal of nonsense
& then announcing that what they just said
makes everything clear

• it is abundantly clear to me that
whatever understanding of math
these students may have had,
it differed sharply from my own;
of course
it could be argued with some force
that their understanding was flawed & partial;
it is a fact that I never faced this kind of situation
with my graduate students;
but then their thinking was evidently closer to my own

☐ randomness

• the question of defining
random/randomness
purely mathematically
remains largely open

• particular notions of random objects
are available in abundance
eg
random event
random number
random sample
random sequence
random variable
random walk
etc;
the task is to determine
the distinguishing & unifying
mathematical characteristic
underlying them all,
if any

• ¿ is randomness capable of being measured ?
¿ physically ?
¿ mathematically ?

• randomness
may not be just two-valued (yes or no)
but subject to measurement
by a real number between 0 and 1 say

• presumably
the notion of
randomness
is tied in with
the notion of
probabilistic phenomenon
&
the mathematical clarification of one
might be equivalent to that of the other

• ¿ does there exist a mathematical definition
of a probabilistic phenomenon
rather than a statistical or philosophical one ?

□ remembering dates

accidental number patterns
may help in remembering
biographical & historical data;
here are some examples

• Euclid
fl ca 300 BCE
mnemonic:
E backwards looks like 3;
¿ how about the zero zero ?
¡ nothing, just nothing, to remember !

• Gauss
1777-1855
mnemonic:
triple prime 7 to double prime 5,
Gauss being great on prime numbers;
in 'great Gauss'
Gg is the 7th letter of the alphabet
&
each word has 5 letters

• Klein
1849-1925
mnemonic:
two squares of primes
$7^2 = 49$ & $5^2 = 25$

• Leibniz
1646-1716
mnemonic:
look at all the powers of 2;
begins & ends with 16;
64 is 'inside' 16
&
71 is 'inside' 16;
646 has internal symmetry

• Newton
1642-1727
mnemonic:
look at the powers of 2 & a power of 3;
begins with a power of 2 & ends with the cube of 3;
727 has internal symmetry;
Newton was born 4 years before Leibniz
& died 11 years after Leibniz

• Newton's 'Principia
was published in 1687
mnemonic:
Newton's age was 45,
giving consecutive integers
4, 5, 6, 7, 8

• Shakespeare
1564-1616
mnemonic:
look at all the powers of 2;
repeat 16, 16;
by long-established tradition
Shakespeare's birthday
is celebrated on 23 April;
his exact birthday is not known for certain;
note the consecutive digits 2, 3, 4
where 4 represents April,
the 4th month of the year;
he died on the same day 23 April
in 1616

• the English mathematician
Augustus De Morgan,
who lived entirely in the 19th century,
said that he was
x years old in the year x^2
whence x = 43
&
he was born in the year
$x^2 - x = 1849 - 43 = 1806$;
he died in 1871;
it was said that he 'read algebra like a novel'

• here are two ways of composing
an elaborate mnemonic for vital dates;
eg consider
Newton
1642-1727

(1) the centuries 16- & 17-
are likely no problem to remember;
for the tens & units,
take two words
with the number positions of the initial letters
in the alphabet
specifying the birth date
&
with the numbers of letters in the words
specifying the death date
eg
'do benefit'
as from
'we do benefit from Newton's mathematics'

(2) compose an appropriate phrase
such that the letter-counts of the consecutive words
gives
the consecutive digits of the vital dates
eg
a genius that is
a benefit to science

brief comment:
sometimes it seems easier just
to remember a few numbers