Extremely Valuable Mean Value Theorems

#43 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2001

GG43-1 (19)

500 Angell St #414
Providence RI 02906

GG43-2

T. Rolle' s Theorem

let

- $a, b \in \mathbb{R}$ st $a < b$
- $f : \mathbb{R}[a, b] \rightarrow \mathbb{R}$
- $f \in$ cont on $\mathbb{R}[a, b]$
- $f \in$ diff on $\mathbb{R}(a, b)$
- $f(a) = f(b)$

then

- $f'(x_0) = 0 \quad (\exists x_0 \in \mathbb{R}(a, b))$

T. RT (verbal version)

consider a real-valued function that is
- defined on a plural bounded closed real interval
- continuous on the closed interval
- differentiable on the open interval
- equal at the endpoints of the interval

then the derivative of the function
vanishes at some point of the open interval

☐ GI of RT

some internal tangent line
is horizontal

T. One - Function Mean Value Theorem for Derivatives

let

- $a, b \in \mathbb{R}$ st $a < b$
- $f : \mathbb{R}[a, b] \to \mathbb{R}$
- $f \in$ cont on $\mathbb{R}[a, b]$
- $f \in$ diff on $\mathbb{R}(a, b)$

then

- $f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$

&

$f(b) - f(a) = (b - a) f'(x_0)$

$(\exists x_0 \in \mathbb{R}(a, b))$

T. 1FMVTD (verbal version)

consider a real-valued function that is
• defined on a plural bounded closed real interval
• continuous on the closed interval
• differentiable on the open interval

then the derivative of the function
at some point of the open interval
equals the slope of the chord
joining the endpoints of the graph of the function

☐ GI of 1FMVTD

some internal tangent line
is parallel to the chord

T. Two - Function Mean Value Theorem for Derivatives

let

- $a, b \in \mathbb{R}$ st $a < b$
- $f, g : \mathbb{R}[a, b] \to \mathbb{R}$
- $f, g \in$ cont on $\mathbb{R}[a, b]$
- $f, g \in$ diff on $\mathbb{R}(a, b)$
- $g(a) \neq g(b)$
- $f'(x) \neq 0 \vee g'(x) \neq 0 \quad (\forall x \in \mathbb{R}(a, b))$

then

- $\dfrac{f'(x_0)}{g'(x_0)} = \dfrac{f(b) - f(a)}{g(b) - g(a)} \quad (\exists x_0 \in \mathbb{R}(a, b))$

T. 2FMVTD (verbal version)

consider two real-valued functions that are
• defined on a plural bounded closed real interval
• continuous on the closed interval
• differentiable on the open interval
and which are such that
• the second function has unequal values
at the endpoints of the interval
• the derivatives of the two functions
do not simultaneously vanish
anywhere in the open interval

then the ratio of the derivatives
at some point of the open interval
equals the ratio of the functional value differences
at the endpoints of the interval

□ GI of 2FMVTD

at some internal point
ratio of derivatives
equals
ratio of slopes of chords

T. One - Function Mean Value Theorem for Integrals

let

- $a, b \in \mathbb{R}$ st $a < b$
- $f : \mathbb{R}[a, b] \to \mathbb{R}$
- $f \in$ cont on $\mathbb{R}[a, b]$

then

- $f(x_0) = \dfrac{1}{b-a}\int_a^b f(x)\,dx$

&

$$\int_a^b f(x)\,dx = (b-a)f(x_0)$$

$(\exists x_0 \in \mathbb{R}(a, b))$

T. 1FMVTI (verbal version)

the definite integral
of a continuous real-valued function
over a plural bounded closed real interval
equals
the length of the interval
times
some internal functional value

☐ statistical interpretation of 1FMVTI

- the average / mean value

of the continuous real - valued function f(x)

on the interval ☼[a, b]

$$=_{df} \quad \frac{1}{b-a}\int_a^b f(x)\,dx$$

- 1FMVTI

$$\frac{1}{b-a}\int_a^b f(x)\,dx \;=\; f(x_0)$$

says that

the function assumes its mean value

at an interior point

□ GI of 1FMVTI

1FMVTI

$$\int_a^b f(x)\,dx = (b-a)\,f(x_0)$$

says that

area under curve

equals

area of rectangle

based on interval

and with some interior ordinate as height

T. Two - Function Mean Value Theorem for Integrals

let

- $a, b \in \mathbb{R}$ st $a < b$
- $f, g : \mathbb{R}[a, b] \to \mathbb{R}$
- $f, g \in$ cont on $\mathbb{R}[a, b]$
- $g \in$ nonneg on $\mathbb{R}[a, b]$

  or

  $g \in$ nonpos on $\mathbb{R}[a, b]$

then

- $\int_a^b f(x)\, g(x)\, dx = f(x_0) \int_a^b g(x)\, dx \quad (\exists x_0 \in \mathbb{R}(a, b))$

T. 2FMVTI (verbal version)

consider two continuous real-valued functions
on a plural bounded closed real interval
with the second function
nonnegative or nonpositive on the interval

then the integral of the product of the functions
over the interval
equals
some interior value of the first function
times
the integral of the second function
over the interval

□ GI of 2FMVTI
is more elaborate

- think of

a 3 - dimensional rectangular xyz - coordinate system

- think of the graph of

$y = f(x),\ a \le x \le b$

as a curve in the xy - plane

- think of the graph of

$z = g(x),\ a \le x \le b$

as a curve in the xz - plane

- this MVT

LHS = RHS

expresses the equality of

the volumes of two solids

- each solid is based on

the region ' under' the curve

$z = g(x), \; a \le x \le b$

and is contained in octant I

- each solid has rectangular cross - sections

perpendicular to the x - axis

with sides in the xy - plane & in the xz - plane

- the solid whose volume is the LHS

has rectangular cross - section

with variable base $g(x), \; a \le x \le b$

and with variable height $f(x), \; a \le x \le b$

- the solid whose volume is the RHS

has rectangular cross - section

with variable base $g(x), \; a \le x \le b$

and with constant height $f(x_0)$