Posets & Tosets & Wosets

#34 of Gottschalk's Gestalts

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of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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□ a quick glance at
the calculus of relations

• X ∈ set;
all considerations are
relative to X;
what ensues is
' a function of X'

• x, y, z ∈ var X

• a relation

$=_{df}$ a set of ordered pairs of elements of X

= a subset of $X \times X$

• the relation space

$=_{dn}$ Rel

$=_{df}$ the set of all relations

= the set of all subsets of $X \times X$

= the power set of $X \times X$

= $\wp(X \times X)$

• R, S $\in$ var Rel

• the universal relation

$=_{dn}$ V

$=_{df}$ $X \times X$

$= \{(x, y) \mid x, y \in X\}$

• the null relation

$=_{dn}$ Λ

$=_{df}$ $\varnothing$

• the identity relation

$=_{dn}$ Δ

$=_{rd}$ del   wh del $\leftarrow$ <u>del</u>ta

$=_{df}$ $\{(x, y) \mid x, y \in X \ \& \ x = y\}$

$=$ $\{(x, x) \mid x \in X\}$

$=$ the diagonal of $X \times X$

• the diversity relation

$=_{dn}$ ∇

$=_{rd}$ codel   wh codel $\leftarrow$ <u>co</u>mplement of <u>del</u>

$=_{df}$ $\{(x, y) \mid x, y \in X \ \& \ x \neq y\}$

$=$ the codiagonal of $X \times X$

• relationship between elements of X:

x bears the relation R to y

= x bears R to y

$=_{dn}$ xRy

$=_{df}$ $(x, y) \in R$

&

$xRySz$

$=_{df}$ $xRy \;\&\; ySz$

etc

• the negation of R

$=_{dn}$ $\not R$

$=_{rd}$ not R

$=_{df}$ $\{(x, y) \mid \neg\, xRy\}$

= the complement of R in $X \times X$

• relational negation

$=_{df}$ the involution of Rel

$R \mapsto \not R$

• the converse of R

$=_{dn}\ \breve{R}$

$=_{rd}$ R con   wh con $\leftarrow$ <u>con</u>verse

$=_{df} \{(x, y) \mid yRx\}$

• relational conversion

$=_{df}$ the involution of Rel

$R \mapsto \breve{R}$

• the product of R and S

$=_{dn}$ RS

$=_{rd}$ R times S

$=_{df}$ $\{(x, z) \mid \exists y.\, xRy \,\&\, ySz\}$

• relational multiplication

$=_{df}$ the binary operation in Rel

$(R, S) \mapsto RS$

• the square of R

$=_{dn}$ R^2

$=_{rd}$ R square(d)

$=_{df}$ RR

• R is reflexive

$=_{df}\ \Delta \subset R$

$\Leftrightarrow\ xRx \quad (\forall x)$

• R is irreflexive

$=_{df}\ R \subset \nabla$

$\Leftrightarrow\ x\not{R}x \quad (\forall x)$

• R is symmetric

$=_{df}\ \breve{R} = R$

$\Leftrightarrow\ xRy \Leftrightarrow yRx \quad (\forall x, y)$

$\Leftrightarrow\ xRy \Rightarrow yRx \quad (\forall x, y)$

• R is asymmetric

$=_{df}\ R \cap \breve{R} = \Lambda$

$\Leftrightarrow\ xRy \Rightarrow y\not{R}x \quad (\forall x, y)$

• R is antisymmetric

$=_{df}\ R \cap \breve{R} \subset \Delta$

$\Leftrightarrow\ xRy\ \&\ yRx \Rightarrow x = y \quad (\forall x, y)$

• R is connected

$=_{df}\ \nabla \subset R \cup \breve{R}$

$\Leftrightarrow\ x \neq y \Rightarrow xRy \vee yRx \quad (\forall x, y)$

• R is transitive

$=_{df}\ R^2 \subset R$

$\Leftrightarrow\ xRy\ \&\ yRz \Rightarrow xRz \quad (\forall x, y, z)$

- the image of x under R

$=$ the projection of x by R

$=_{dn}$ xR

$=_{df} \{y \mid xRy\}$

- $A \subset X \Rightarrow$

the image of A under R

$=$ the projection of A by R

$=_{dn}$ AR

$=_{df} \bigcup \{xR \mid x \in A\}$

- relational projection

$=_{df}$ the function $\wp X \times \mathrm{Rel} \to \wp X$

$(A, R) \mapsto AR$

• the domain of R

$=_{dn}$ dmn R

$=_{df}$ $X\breve{R}$

$= \{x \mid \exists y \,.\, xRy\}$

• the range of R

$=_{dn}$ rng R

$=_{df}$ XR

$= \{y \mid \exists x \,.\, xRy\}$

• the field of R

$=_{dn}$ fld R

$=_{df}$ dmn R $\cup$ rng R

note: the abbreviating notation

dmn, rng, fld

consists of the consonants in the words

domain, range, field

• an R - descending sequence

$=_{df}$ a sequence

$x_1, x_2, x_3, \cdots$

of pairwise distinct elements of fld R st

$x_1 \breve{R} x_2 \breve{R} x_3 \breve{R} \cdots$

which may be written

$\cdots R x_3 R x_2 R x_1$

• an R - ascending sequence

$=_{df}$ a sequence

$x_1, x_2, x_3, \cdots$

of pairwise distinct elements of fld R st

$x_1 R x_2 R x_3 R \cdots$

• R is left - founded

$=_{df}$ there does not exist an R - descending sequence

• R is right - founded

$=_{df}$ there does not exist an R - ascending sequence

note: here ' founded' has the meaning
' has a base / foundation'

□ the notion & notation of order in a set

• in defining
an ordered set
or
a partially ordered set
or
a totally ordered set
etc
whether in general or in particular
one is faced with the choice of notation
viz
to use $\leq$ or $\geq$ or $<$ or $>$ as the initial defining order relation;
the notion of wedge
(just take all four of them)
overcomes that notational awkwardness;
now define the however ordered set as
a set provided with a wedge
& then make the usual statements;
in this way all four relations have equal status
right from the beginning
& one can choose in definition or development
whatever relation happens to be the most convenient;
¿ why the word 'wedge' ?
each of the four inequality signs
$\leq , \geq , < , >$
contains or is a wedge-shaped mark;
so the four altogether constitute
a packet of wedges = a wedge pack
or simply 'wedge' for short;

each of the four relations

≤, ≥, <, >

in a wedge
should uniquely determine the other three;
if this customary notation for order is used,
then the following should be the case:

the notation ≤ or ≥ should be used for a relation

only if the relation is reflexive at least
&
the notation < or > should be used for a relation
only if the relation is irreflexive at least;

≤ should mean < or =

&

≥ should mean > or =

&

< should mean ≤ and ≠

&

> should mean ≥ and ≠ ;

≤ and ≥ should be converses of each other;

< and > should be converses of each other;
these considerations
lead to the notion of wedge

D. wedges

let

X $\in$ set

then

a wedge of X

$=_{df}$ an ordered pair of ordered pairs of relations in X

$=_{dn}$ $((\leq, <), (\geq, >))$ canonically

$=_{ab}$ $(\leq, <; \geq, >)$

st

- $\breve{\leq} = \geq$ wiet $\breve{\geq} = \leq$
- $\breve{<} = >$ wiet $\breve{>} = <$
- $\leq \cap \nabla = <$ & $\leq = < \cup \Delta$
- $\geq \cap \nabla = >$ & $\geq = > \cup \Delta$

(some redundancy here)

D. wedged sets

• a wedged set
$=_{df}$ a set X equipped with
a wedge $(\leq, <; \geq, >)$ of X
ie
an ordered pair
$(X, (\leq, <; \geq, >))$
consisting of a set X
and a wedge $(\leq, <; \geq, >)$ of X;
notationally
it is customary to call X alone the wedged set
rather than to call the ordered pair only the wedged set,
the order notation being understood;
such an abbreviating notational device
occurs frequently in mathematics

R. every duality is based upon an involution;
the following duality
is based upon relational conversion

T. law of duality for wedges
let
• $X \in$ set
• $\leq, \; <, \; \geq, \; > \; \in$ rel in X

then
$(\leq, \; < ; \geq, \; >) \in$ wedge of X
$\Leftrightarrow$
$(\geq, \; > ; \leq, \; <) \in$ wedge of X

D. the above observation leads to
the following definitions:
• the dual of the wedge $(\leq, \; < ; \geq, \; >)$ of a set X
$=_{df}$ the wedge $(\geq, \; > ; \leq, \; <)$ of X
• the dual of the wedged set $(X, \; (\leq, \; < ; \geq, \; >))$
$=_{df}$ the wedged set $(X, \; (\geq, \; > ; \leq, \; <))$

D. terminology for wedged sets

let

- $X \in$ wedged set
- $x, y \in X$

then

- $\leq$ $=_{cl}$ the weak lohi inequality in X
- $<$ $=_{cl}$ the strict lohi inequality in X
- $\geq$ $=_{cl}$ the weak hilo inequality in X
- $>$ $=_{cl}$ the strict hilo inequality in X

- $x \leq y$ $=_{rd}$ x is weakly less than y
- $x < y$ $=_{rd}$ x is strictly less than y
- $x \geq y$ $=_{rd}$ x is weakly greater than y
- $x > y$ $=_{rd}$ x is strictly greater than y

syntactically

- $\leq$ $=_{cl}$ the weak lohi inequality sign
- $<$ $=_{cl}$ the strict lohi inequality sign
- $\geq$ $=_{cl}$ the weak hilo inequality sign
- $>$ $=_{cl}$ the strict hilo inequality sign

- $\nleq$ $=_{cl}$ the negated weak lohi inequality in X
- $\nless$ $=_{cl}$ the negated strict lohi inequality in X
- $\ngeq$ $=_{cl}$ the negated weak hilo inequality in X
- $\ngtr$ $=_{cl}$ the negated strict hilo inequality in X

- $x \nleq y$ $=_{rd}$ x is not weakly less than y
- $x \nless y$ $=_{rd}$ x is not strictly less than y
- $x \ngeq y$ $=_{rd}$ x is not weakly greater than y
- $x \ngtr y$ $=_{rd}$ x is not strictly greater than y

syntactically

- $\nleq$ $=_{cl}$ the negated weak lohi inequality sign
- $\nless$ $=_{cl}$ the negated strict lohi inequality sign
- $\ngeq$ $=_{cl}$ the negated weak hilo inequality sign
- $\ngtr$ $=_{cl}$ the negated strict hilo inequality sign

note:

lohi ← low high

hilo ← high low

R. let

• $X \in$ wedged set

then

• $\leq$ is reflexive in X

ie

$x \leq x \quad (\forall x \in X)$

• $\geq$ is reflexive in X

ie

$x \geq x \quad (\forall x \in X)$

• $<$ is irreflexive in X

ie

$x \nless x \quad (\forall x \in X)$

• $>$ is irreflexive in X

ie

$x \ngtr x \quad (\forall x \in X)$

R. let

- $X \in$ set
- $R \in$ reflexive relation in X
- $\leq \;=_{df}\; R$
- $< \;=_{df}\; R \cap \nabla$
- $\geq \;=_{df}\; \breve{R}$
- $> \;=_{df}\; \breve{R} \cap \nabla$

then

- $(\leq,\ <; \geq,\ >) \in$ wedge of X

R. let

- $X \in$ set
- $R \in$ reflexive relation in X

then

- there exists exactly one wedge $(\leq,\ <; \geq,\ >)$ of X

st $\leq \;=\; R$

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R. let

- $X \in$ set
- $R \in$ reflexive relation in X
- $\geq \ =_{df} \ R$
- $> \ =_{df} \ R \cap \nabla$
- $\leq \ =_{df} \ \breve{R}$
- $< \ =_{df} \ \breve{R} \cap \nabla$

then

- $(\leq, \ <; \geq, \ >) \in$ wedge of X

R. let

- $X \in$ set
- $R \in$ reflexive relation in X

then

- there exists exactly one wedge $(\leq, \ <; \geq, \ >)$ of X

st $\geq \ = \ R$

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R. let

- $X \in$ set
- $R \in$ irreflexive relation in X
- $< \; =_{df} \; R$
- $\leq \; =_{df} \; R \cup \Delta$
- $> \; =_{df} \; \breve{R}$
- $\geq \; =_{df} \; \breve{R} \cup \Delta$

then

- $(\leq, \; <; \geq, \; >) \in$ wedge of X

R. let

- $X \in$ set
- $R \in$ irreflexive relation in X

then

- there exists exactly one wedge $(\leq, \; <; \geq, \; >)$ of X

st $< \; = \; R$

R. let

- $X \in$ set
- $R \in$ irreflexive relation in X
- $> \; =_{df} \; R$
- $\geq \; =_{df} \; R \cup \Delta$
- $< \; =_{df} \; \breve{R}$
- $\leq \; =_{df} \; \breve{R} \cup \Delta$

then

- $(\leq, \; <; \geq, \; >) \in$ wedge of X

R. let

- $X \in$ set
- $R \in$ irreflexive relation in X

then

- there exists exactly one wedge $(\leq, \; <; \geq, \; >)$ of X

st $> \; = \; R$

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D. intervals

let

- $X \in$ wedged set
- $a, b \in X$

then

- the closed interval of X from a to b

$=_{dn}$ $X[a,b]$

$=_{df}$ $\{x \mid x \in X \ \& \ a \le x \le b\}$

& ∴

$x \in X[a,b] \Leftrightarrow x \in X \ \& \ a \le x \le b$

- the open interval of X from a to b

$=_{dn}$ $X(a,b)$

$=_{df}$ $\{x \mid x \in X \ \& \ a < x < b\}$

& ∴

$x \in X(a,b) \Leftrightarrow x \in X \ \& \ a < x < b$

• the left - closed right - open interval of X from a to b

$=_{dn}$ $X[a, b)$

$=_{df}$ $\{x \mid x \in X \;\&\; a \le x < b\}$

& ∴

$x \in X[a, b) \;\Leftrightarrow\; x \in X \;\&\; a \le x < b$

• the left - open right - closed interval of X from a to b

$=_{dn}$ $X(a, b]$

$=_{df}$ $\{x \mid x \in X \;\&\; a < x \le b\}$

& ∴

$x \in X(a, b] \;\Leftrightarrow\; x \in X \;\&\; a < x \le b$

• a half - closed / half - open / semi - closed / semi - open interval of X from a to b

$=_{df}$ $X[a, b)$ or $X(a, b]$

D. points relating to intervals

let

- $X \in$ wedged set
- $a, b \in X$
- I stands for any one of the intervals

$X[a, b]$

$X(a, b)$

$X[a, b)$

$X(a, b]$

then

- an endpoint of I

$=_{df}$ a or b

- the left / initial endpoint of I

$=_{df}$ a

- the right / terminal endpoint of I

$=_{df}$ b

- the boundary of I

$=_{df} \{a, b\}$

• an interior point of I

= an inpoint of I

= a point that is interior to I

$=_{df}$ an element of X(a, b)

• the interior of I

$=_{df}$ X(a, b)

• an exterior point of I

= an expoint of I

= a point that is exterior to I

$=_{df}$ an element of $X(\leftarrow, a) \cup X(b, \rightarrow)$

• the exterior of I

$=_{df}$ $X(\leftarrow, a) \cup X(b, \rightarrow)$

D. rays

let

• $X \in$ wedged set

• $a \in X$

then

• the closed right ray of X from a

$=_{dn}$ $X[a, \rightarrow)$

$=_{df}$ $\{x \mid x \in X \ \& \ a \leq x\}$

& $\therefore$

$x \in X[a, \rightarrow) \iff x \in X \ \& \ a \leq x$

• the open right ray of X from a

$=_{dn}$ $X(a, \rightarrow)$

$=_{df}$ $\{x \mid x \in X \ \& \ a < x\}$

& $\therefore$

$x \in X(a, \rightarrow) \iff x \in X \ \& \ a < x$

• the closed left ray of X from a

$=_{dn}$ $X(\leftarrow, a]$

$=_{df}$ $\{x \mid x \in X \;\&\; x \leq a\}$

& ∴

$x \in X(\leftarrow, a] \Leftrightarrow x \in X \;\&\; x \leq a$

• the open left ray of X from a

$=_{dn}$ $X(\leftarrow, a)$

$=_{df}$ $\{x \mid x \in X \;\&\; x < a\}$

& ∴

$x \in X(\leftarrow, a) \Leftrightarrow x \in X \;\&\; x < a$

D. points relating to rays

let

• $X \in$ wedged set

• $a \in X$

then

• the endpoint / vertex / boundary of

$X[a, \rightarrow)$

or

$X(a, \rightarrow)$

or

$X(\leftarrow, a]$

or

$X(\leftarrow, a)$

$=_{df}$ a

• an interior point of $X[a, \rightarrow)$ or $X(a, \rightarrow)$
= an inpoint of $X[a, \rightarrow)$ or $X(a, \rightarrow)$
= a point that is interior to $X[a, \rightarrow)$ or $X(a, \rightarrow)$
$=_{df}$ an element of $X(a, \rightarrow)$

• the interior of $X[a, \rightarrow)$ or $X(a, \rightarrow)$
$=_{df}$ $X(a, \rightarrow)$

• an exterior point of $X[a, \rightarrow)$ or $X(a, \rightarrow)$
= an expoint of $X[a, \rightarrow)$ or $X(a, \rightarrow)$
= a point that is exterior to $X[a, \rightarrow)$ or $X(a, \rightarrow)$
$=_{df}$ an element of $X(\leftarrow, a)$

• the exterior of $X[a, \rightarrow)$ or $X(a, \rightarrow)$
$=_{df}$ $X(\leftarrow, a)$

• an interior point of $X(\leftarrow, a]$ or $X(\leftarrow, a)$
= an inpoint of $X(\leftarrow, a]$ or $X(\leftarrow, a)$
= a point that is interior to $X(\leftarrow, a]$ or $X(\leftarrow, a)$
$=_{df}$ an element of $X(\leftarrow, a)$

• the interior of $X(\leftarrow, a]$ or $X(\leftarrow, a)$
$=_{df}$ $X(\leftarrow, a)$

• an exterior point of $X(\leftarrow, a]$ or $X(\leftarrow, a)$
= an expoint of $X(\leftarrow, a]$ or $X(\leftarrow, a)$
= a point that is exterior to $X(\leftarrow, a]$ or $X(\leftarrow, a)$
$=_{df}$ an element of $X(a, \rightarrow)$

• the exterior of $X(\leftarrow, a]$ or $X(\leftarrow, a)$
$=_{df}$ $X(a, \rightarrow)$

D. partially ordered sets

a partially ordered set
$=_{ab}$ poset
$=_{df}$ a wedged set X st
• $\leq$ is a reflexive antisymmetric transitive relation in X
ie
• these proprties hold:
reflexivity of $\leq$ in X
$x \leq x \quad (x \in X)$
&
antisymmetry of $\leq$ in X
$x \leq y \ \& \ y \leq x \Rightarrow x = y \quad (x, y \in X)$
&
transitivity of $\leq$ in X
$x \leq y \ \& \ y \leq z \Rightarrow x \leq z \quad (x, y, z \in X)$

wiet

• $\geq$ is a reflexive antisymmetric transitive relation in X

ie

• these properties hold:

reflexivity of $\geq$ in X

$x \geq x \quad (x \in X)$

&

antisymmetry of $\geq$ in X

$x \geq y \ \& \ y \geq x \ \Rightarrow \ x = y \quad (x, y \in X)$

&

transitivity of $\geq$ in X

$x \geq y \ \& \ y \geq z \ \Rightarrow \ x \geq z \quad (x, y, z \in X)$

wiet

• $<$ is an irreflexive transitive relation in X

ie

• these properties hold:

irreflexivity of $<$ in X

$x \not< x \quad (x \in X)$

&

transitivity of $<$ in X

$x < y \;\&\; y < z \;\Rightarrow\; x < z \quad (x, y, z \in X)$

wiet

• $>$ is an irreflexive transitive relation in X

ie

• these properties hold:

irreflexivity of $>$ in X

$x \not> x \quad (x \in X)$

&

transitivity of $>$ in X

$x > y \ \& \ y > z \Rightarrow x > z \quad (x, y, z \in X)$

C. the mention of reflexivity & irreflexivity
in the preceding definition is logically correct
but unnecessary
since the component relations of a wedge
have these properties automatically;
however, the following considerations show that
the inclusion is desirable for an easier overview
of the basic notions

D. the following definitions
are essentially equivalent:
$X \in$ set
$\Rightarrow$
• a partial order in X
$=_{df}$ a reflexive antisymmetric transitive
relation in X
where the notation $\leq$ or $\geq$ is used for this relation
&
• a partial order in X
$=_{df}$ an irreflexive transitive
relation in X
where the notation $<$ or $>$ is used for this relation

D. the following definition
is essentially equivalent to
the preceding definition of poset
as a wedged set of a certain kind:
• a partially ordered set
$=_{df}$ a set X equipped with a partial order in X

D. comparable elements

let

- $X \in$ wedged set
- $x, y \in X$

then

- x is comparable to / with y

= x and y are comparable

$=_{dn}$ $x \sim y$

$=_{rd}$ x comp y

$=_{df}$ $x \neq y \Rightarrow x < y \vee x > y$

$\Leftrightarrow x < y \vee x = y \vee x > y$

$\Leftrightarrow x \leq y \vee x > y$

$\Leftrightarrow x < y \vee x \geq y$

$\Leftrightarrow x \leq y \vee x \geq y$

- x is incomparable to / with y

= x and y are incomparable

$=_{dn}$ $x \nsim y$

$=_{rd}$ x incomp y

$=_{df}$ $\neg x \sim y$

R . connectivity

let

- X $\in$ wedged set

then

tfsape

- every pair of elements of X are comparable
- $\leq$ is connected in X
- $\geq$ is connected in X
- $<$ is connected in X
- $>$ is connected in X

D. totally ordered sets

a totally ordered set
$=_{ab}$ toset
$=_{df}$ a wedged set X st
• $\leq$ is a reflexive antisymmetric transitive connected relation in X
ie
• these proprties hold:
reflexivity of $\leq$ in X
$x \leq x \quad (x \in X)$
&
antisymmetry of $\leq$ in X
$x \leq y \ \& \ y \leq x \ \Rightarrow \ x = y \quad (x, y \in X)$
&
transitivity of $\leq$ in X
$x \leq y \ \& \ y \leq z \ \Rightarrow \ x \leq z \quad (x, y, z \in X)$
&
connectivity of $\leq$ in X
$x \neq y \ \Rightarrow \ x \leq y \vee y \leq x \quad (x, y \in X)$

wiet

• $\geq$ is a reflexive antisymmetric transitive connected relation in X

ie

• these properties hold:

reflexivity of $\geq$ in X

$x \geq x \quad (x \in X)$

&

antisymmetry of $\geq$ in X

$x \geq y \;\&\; y \geq x \;\Rightarrow\; x = y \quad (x, y \in X)$

&

transitivity of $\geq$ in X

$x \geq y \;\&\; y \geq z \;\Rightarrow\; x \geq z \quad (x, y, z \in X)$

&

connectivity of $\geq$ in X

$x \neq y \;\Rightarrow\; x \geq y \vee y \geq x \quad (x, y \in X)$

wiet

• $<$ is an irreflexive transitive connected relation in X

ie

• these proprties hold:

irreflexivity of $<$ in X

$x \nless x \quad (x \in X)$

&

transitivity of $<$ in X

$x < y \ \& \ y < z \Rightarrow x < z \quad (x, y, z \in X)$

&

connectivity of $<$ in X

$x \neq y \Rightarrow x < y \vee y < x \quad (x, y \in X)$

wiet

• $>$ is an irreflexive transitive connected

relation in X

ie

• these proprties hold:

irreflexivity of $>$ in X

$x \not> x \quad (x \in X)$

&

transitivity of $>$ in X

$x > y \;\&\; y > z \;\Rightarrow\; x > z \quad (x, y, z \in X)$

&

connectivity of $>$ in X

$x \neq y \;\Rightarrow\; x > y \vee y > x \quad (x, y \in X)$

C. the mention of reflexivity & irreflexivity
in the preceding definition is logically correct
but unnecessary
since the component relations of a wedge
have these properties automatically;
however, the following considerations show that
the inclusion is desirable for an easier overview
of the basic notions

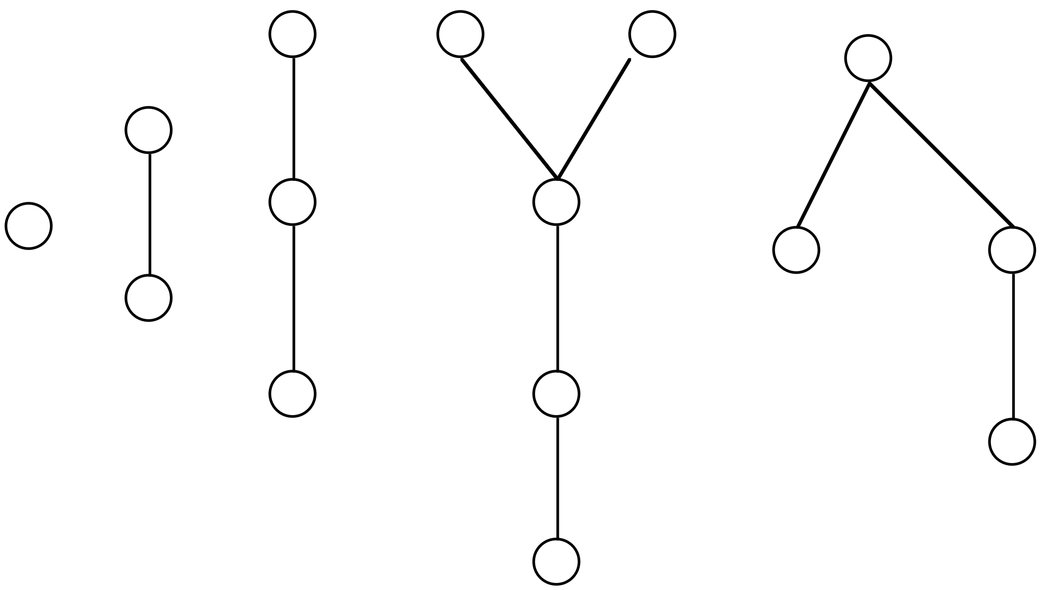
D. the following definitions
are essentially equivalent:
X ∈ set
⇒
• a total order in X
$=_{df}$ a reflexive antisymmetric transitive connected relation in X
where the notation $\leq$ or $\geq$ is used for this relation
&
• a total order in X
$=_{df}$ an irreflexive transitive connected relation in X
where the notation $<$ or $>$ is used for this relation

D. the following definition
is essentially equivalent to
the preceding definition of toset
as a wedged set of a certain kind:
• a totally ordered set
$=_{df}$ a set X equipped with a total order in X

T. laws of duality
• the dual of a poset is a poset
• the dual of a toset is a toset

C. a finite poset may often be visualized
as a display of small circles in the plane
that represent the elements of the poset
and in which two circles are connected by
a line segment (thought of as rising)
if the lower is strictly less than the upper
and there is no element strictly inbetween;
one element is strictly less than another
iff there is a rising path from one to the other;
in this pictorialization
to dualize a poset = to pass to its dual
is then to turn the display upside down;
often a good geometric image of a toset
is a piece of a straight line,
horizontal or vertical;
the following Geometric Pictures
make these ideas more concrete

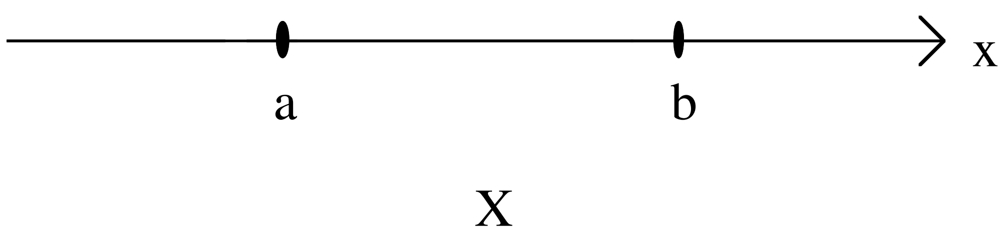
GP. visualization of a finite poset



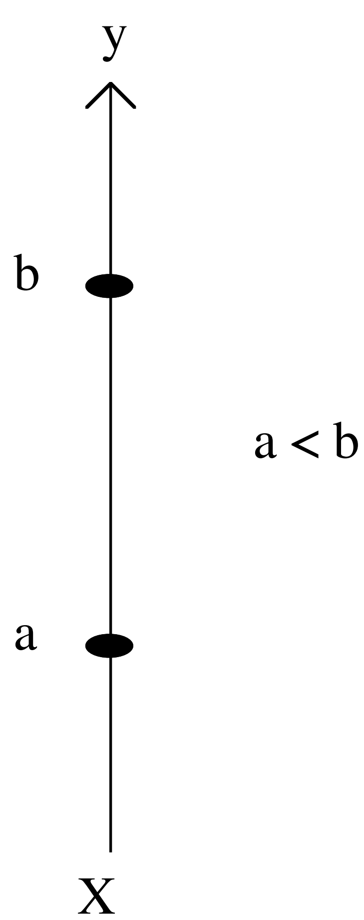
GP. a toset may be visualized as all of or a part of
a directed horizontal line
or
a directed vertical line
whose points are the elements of the toset

• a toset X visualized as all or part of
a directed horizontal line
like the x - axis in a customarily drawn
rectangular coordinate system
where the specified positive direction of the x - axis
is from left to right

$a < b$



• a toset X visualized as all or part of
a directed vertical line
like the y - axis in a customarily drawn
rectangular coordinate system
where the specified positive direction of the y - axis
is from below to above



GG34-55

C. these last two pictures
help account for the order-theoretic terminology
from the horizntal POV
as
left, right; preceding, succeeding; etc
&
from the vertical POV
as
lower, upper; below, above; etc

N. synonyms

• order (noun)
= ordering

• total order
= simple order
= linear order
= order

• totally ordered set
= simply ordered set
= linearly ordered set
= ordered set

• less / lesser = smaller

• greater = larger

• least = smallest

• greatest = largest

C. ¿ in or out ?

let A be an interval or a ray of a wedged set;

then

• an endpoint of A is or is not an element of A
according as it is
' at a closed end'
or
' at an open end'
of A

• an interior point of A is always an element of A

• an exterior point of A is never an element of A
if X is a poset

C. the nature of endpoints:
it may occur that
an endpoint of an interval or ray
is not uniquely determined by
the interval or ray as a set;
in which case the endpoint
is to be regarded as a ' formal' object
ie
an object specified by
the formal expression that names the set
and not
an object specified by
the set that the formal expression names

N. the notation for intervals and rays
may often be streamlined
by dropping the sign for the wedged set;
this may occur when no ambiguity is introduced;
however,
when an open interval is considered
the new notation becomes
the same as that for an ordered pair;
caution must be observed in letting
the context determine the meaning;
thus for a wedged set X with $a, b \in X$;

$X[a,b] = [a,b]$

$X(a,b) = (a,b)$

$X[a,b) = [a,b)$

$X(a,b] = (a,b]$

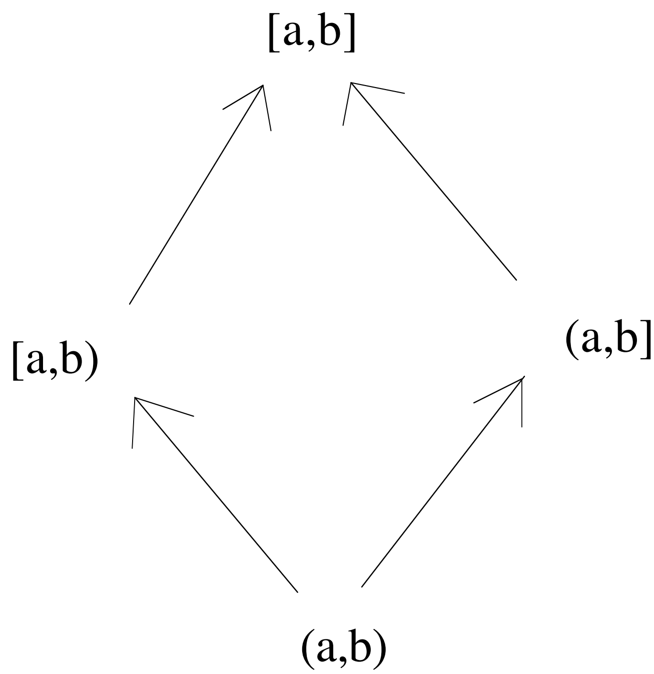
$X[a,\rightarrow) = [a,\rightarrow)$

$X(a,\rightarrow) = (a,\rightarrow)$

$X(\leftarrow,a] = (\leftarrow,a]$

$X(\leftarrow,a) = (\leftarrow,a)$

R. the little lattice of intervals
of a wedged set



lattice inclusion = set inclusion
meet = intersection
join = union

$[a, b] \ = \ [a, b) \ \cup \ (a, b]$

$(a, b) \ = \ [a, b) \ \cap \ (a, b]$

GG34-61

R. every interval is the intersection of two rays

viz

$a, b \in$ wedged set $\Rightarrow$

$$[a,b] = [a,\rightarrow) \cap (\leftarrow,b]$$

$$(a,b) = (a,\rightarrow) \cap (\leftarrow,b)$$

$$[a,b) = [a,\rightarrow) \cap (\leftarrow,b)$$

$$(a,b] = (a,\rightarrow) \cap (\leftarrow,b]$$

diagrammatically

$$\begin{array}{ccccccc}
 & & & [a,b] & & & \\
 & & & \| & & & \\
 & & [a,\rightarrow) & \cap & (\leftarrow,b] & & \\
[a,b) & = & \cap & & \cap & = & (a,b] \\
 & & (\leftarrow,b) & \cap & (a,\rightarrow) & & \\
 & & & \| & & & \\
 & & & (a,b) & & &
\end{array}$$

rays intersect in intervals

R. intersection & union of rays
with the same endpoint:

$a \in$ toset $X \Rightarrow$

$(\leftarrow, a] \cap [a, \rightarrow) = \{a\}$

$(\leftarrow, a] \cap (a, \rightarrow) = \varnothing$

$(\leftarrow, a) \cap [a, \rightarrow) = \varnothing$

$(\leftarrow, a) \cap (a, \rightarrow) = \varnothing$

$(\leftarrow, a] \cup [a, \rightarrow) = X$

$(\leftarrow, a] \cup (a, \rightarrow) = X$

$(\leftarrow, a) \cup [a, \rightarrow) = X$

$(\leftarrow, a) \cup (a, \rightarrow) = X - \{a\}$

R. decompositions of a toset into intervals & rays:

$a, b \in$ toset X with $a \leq b \Rightarrow$

X

$= (\leftarrow, a) \dot{\cup} [a, b] \dot{\cup} (b, \rightarrow)$

$= (\leftarrow, a] \dot{\cup} (a, b) \dot{\cup} [b, \rightarrow)$

$= (\leftarrow, a) \dot{\cup} [a, b) \dot{\cup} [b, \rightarrow)$

$= (\leftarrow, a] \dot{\cup} (a, b] \dot{\cup} (b, \rightarrow)$

$= (\leftarrow, a] \dot{\cup} (a, \rightarrow)$

$= (\leftarrow, a) \dot{\cup} [a, \rightarrow)$

D. the closed / open unit intervals

• the closed unit interval

= the unit interval

$=_{dn}$ $\mathbb{I}$

$=_{rd}$ (cap open) eye

$=_{df}$ $\mathbb{R}[0,1]$

$= \{x \mid x \in \text{real nr} \;\&\; 0 \leq x \leq 1\}$

• the open unit interval

$=_{dn}$ $\overset{o}{\mathbb{I}}$

$=_{rd}$ (cap open) eye (overscript / over) oh

$=_{df}$ $\mathbb{R}(0,1)$

$= \{x \mid x \in \text{real nr} \;\&\; 0 < x < 1\}$

note: the small circle in $\overset{o}{\mathbb{I}}$
is the interior operator

N. paraphrases

• wedge of X
= wedge for X
= wedge in X

• interval of X
= interval in X

• ray of X
= ray in X

• admissible order of X
= admissible order for X
= admissible order in X

• comparable to
= comparable with

GP. geometric pictures of intervals & rays
at an exhibition of the real line

• the following diagrams illustrate
the eight kinds of intervals & rays
in the real number set ℝ
with its natural total order
by appealing to its geometric analog, the straight line;
both the ' horizontal' & ' vertical' interpretations
are given

r is below a real number variable

• the closed interval of $\mathbb{R}$

from a to b

wh $a, b \in \mathbb{R}$ st $a < b$

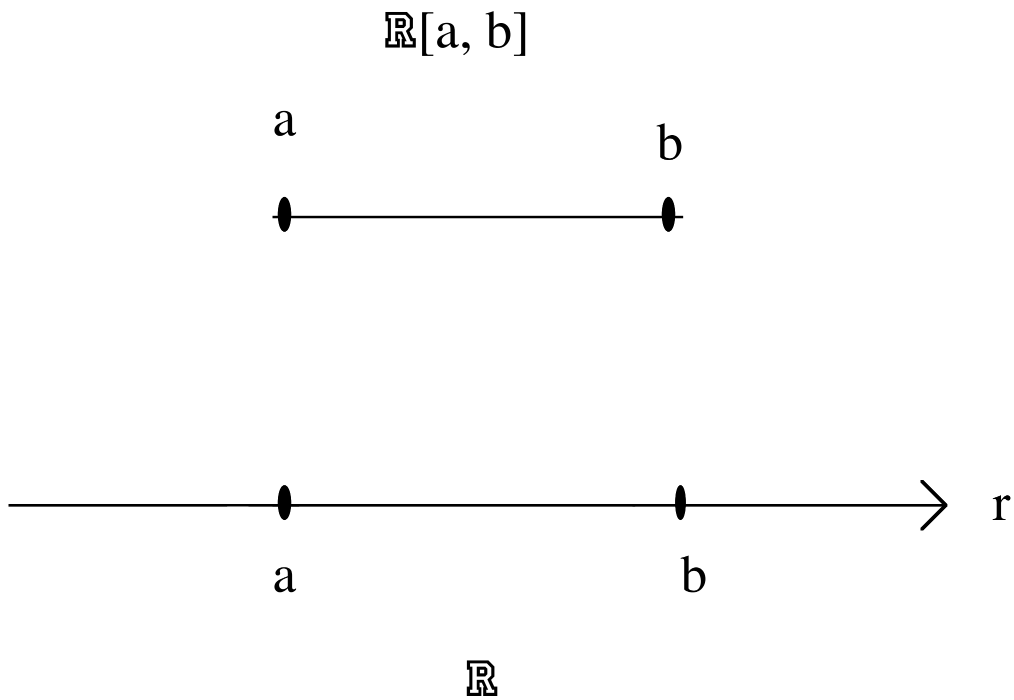
$= \{r \mid r \in \mathbb{R} \ \& \ a \leq r \leq b\}$

$= \mathbb{R}[a, b]$

&

$r \in \mathbb{R}[a, b] \Leftrightarrow r \in \mathbb{R} \ \& \ a \leq r \leq b$

pictured horizontally



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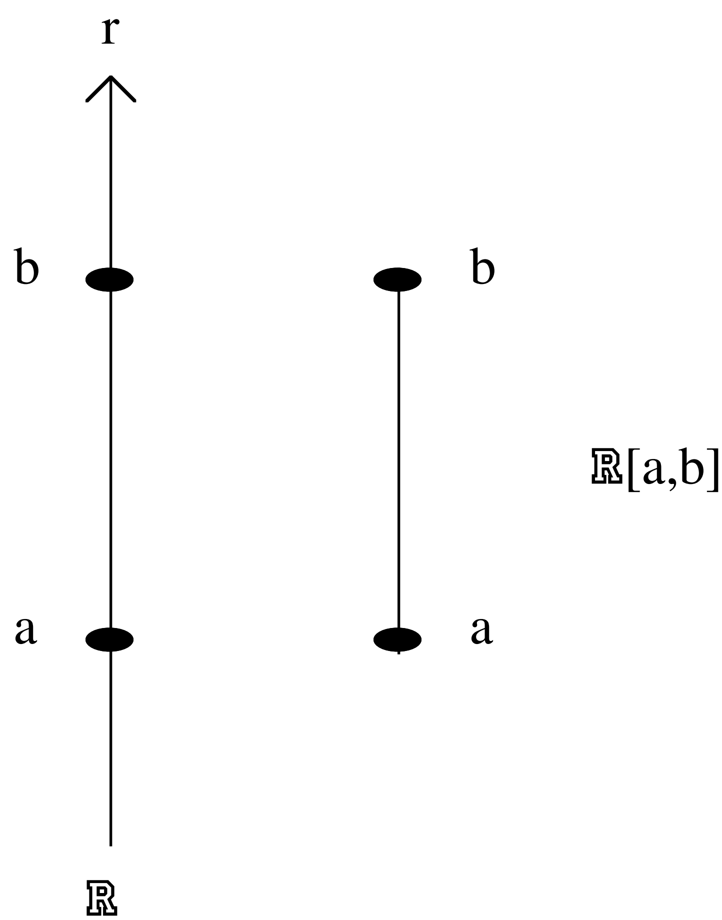


GG34-70

• the open interval of $\mathbb{R}$

from a to b

wh $a, b \in \mathbb{R}$ st $a < b$

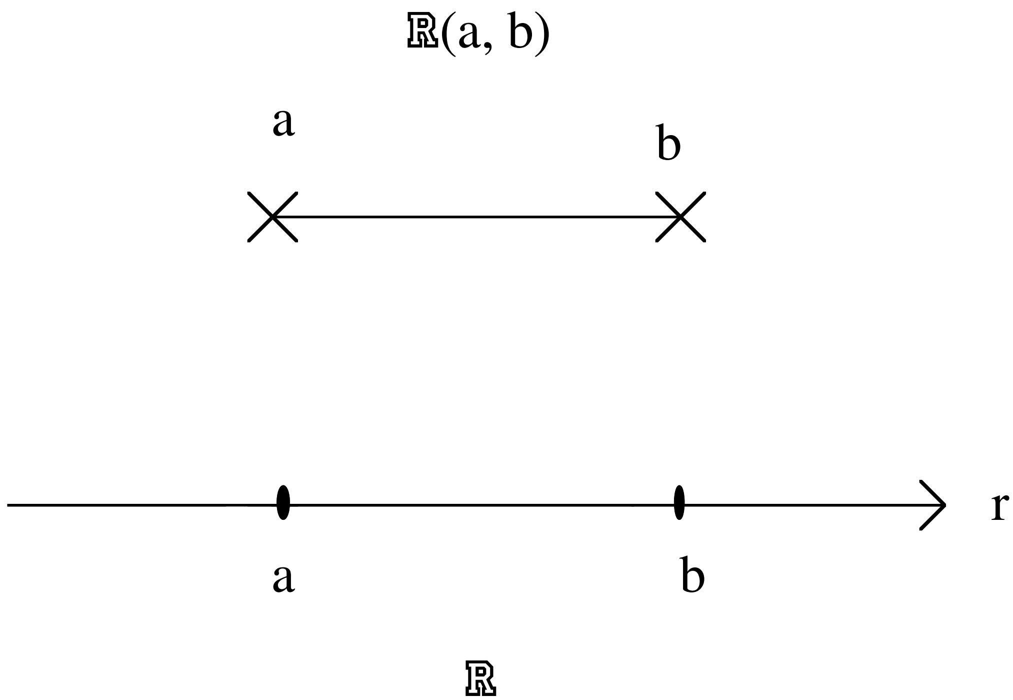
$= \{r \mid r \in \mathbb{R} \ \& \ a < r < b\}$

$= \mathbb{R}(a, b)$

&

$r \in \mathbb{R}(a, b) \Leftrightarrow r \in \mathbb{R} \ \& \ a < r < b$

pictured horizontally



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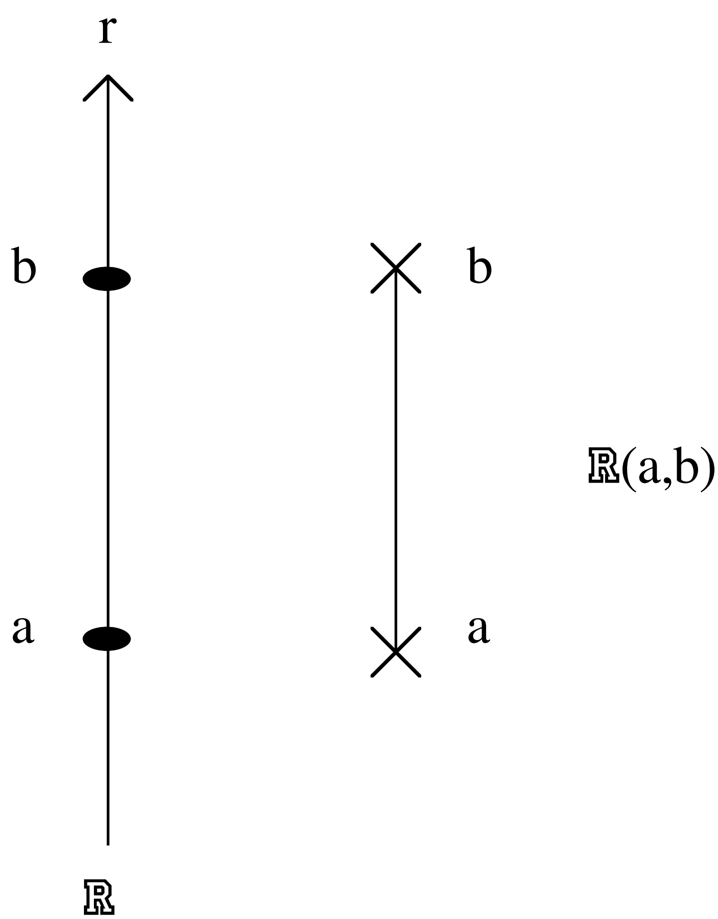
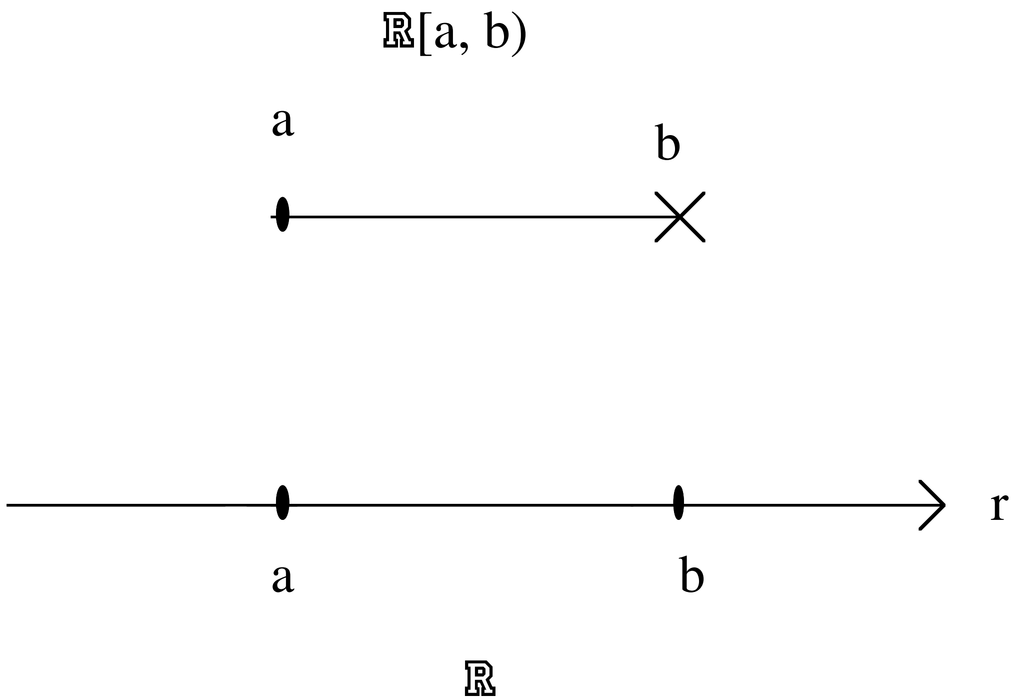


• the left - closed right - open interval of $\mathbb{R}$
from a to b
wh $a, b \in \mathbb{R}$ st $a < b$
$= \{r \mid r \in \mathbb{R} \ \& \ a \leq r < b\}$
$= \mathbb{R}[a, b)$
&
$r \in \mathbb{R}[a, b) \Leftrightarrow r \in \mathbb{R} \ \& \ a \leq r < b$

pictured horizontally



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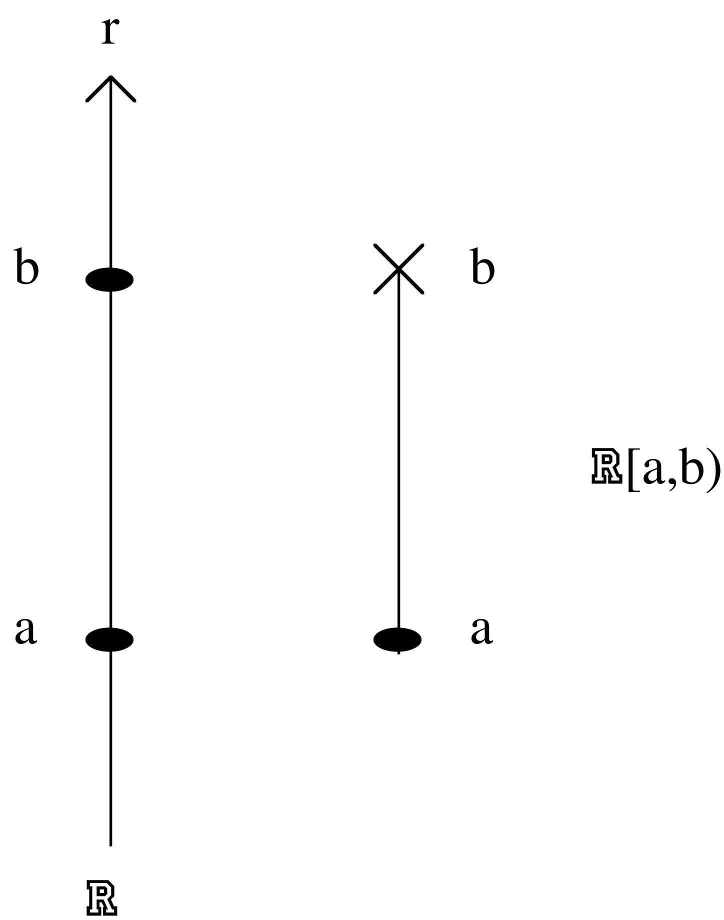


GG34-76

• the left - open right - closed interval of $\mathbb{R}$

from a to b

wh $a, b \in \mathbb{R}$ st $a < b$

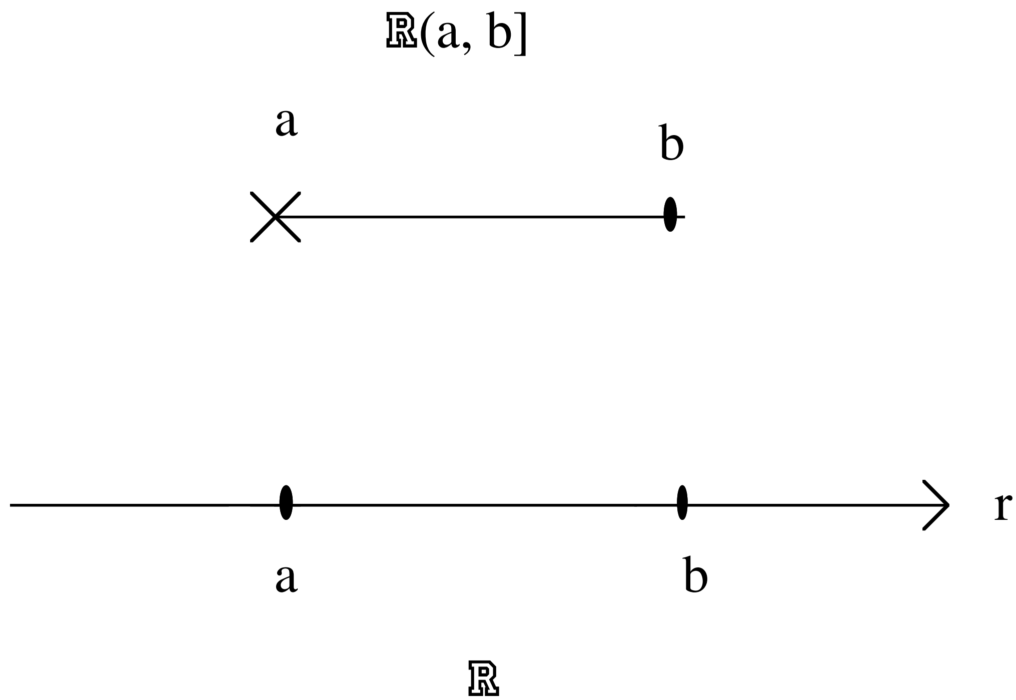
$= \{r \mid r \in \mathbb{R} \ \& \ a < r \leq b\}$

$= \mathbb{R}(a, b]$

&

$r \in \mathbb{R}(a, b] \Leftrightarrow r \in \mathbb{R} \ \& \ a < r \leq b$

pictured horizontally



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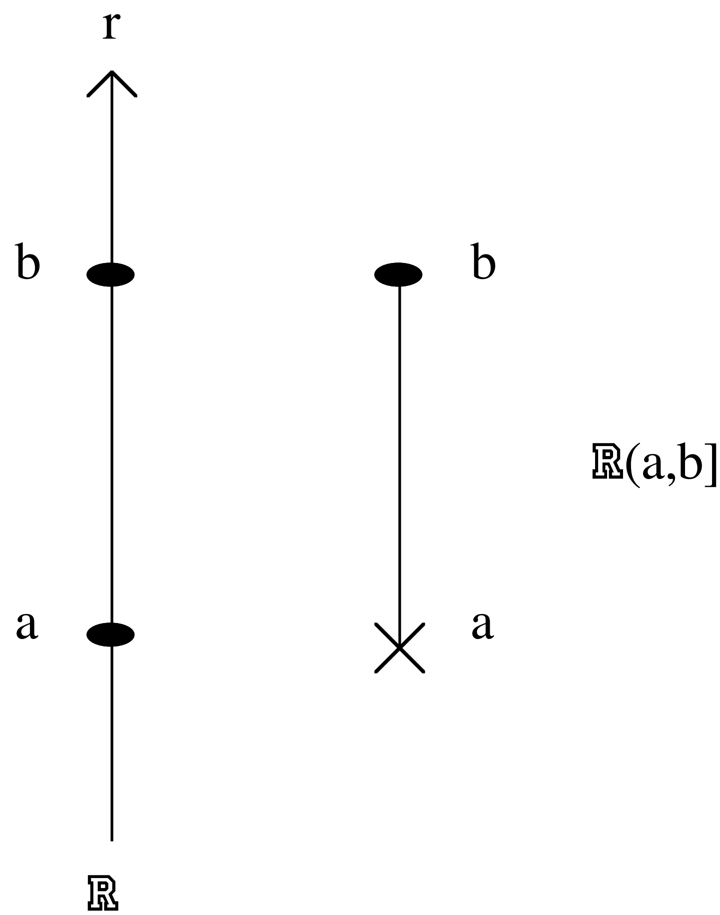


• the closed right ray of $\mathbb{R}$

from a

wh $a \in \mathbb{R}$

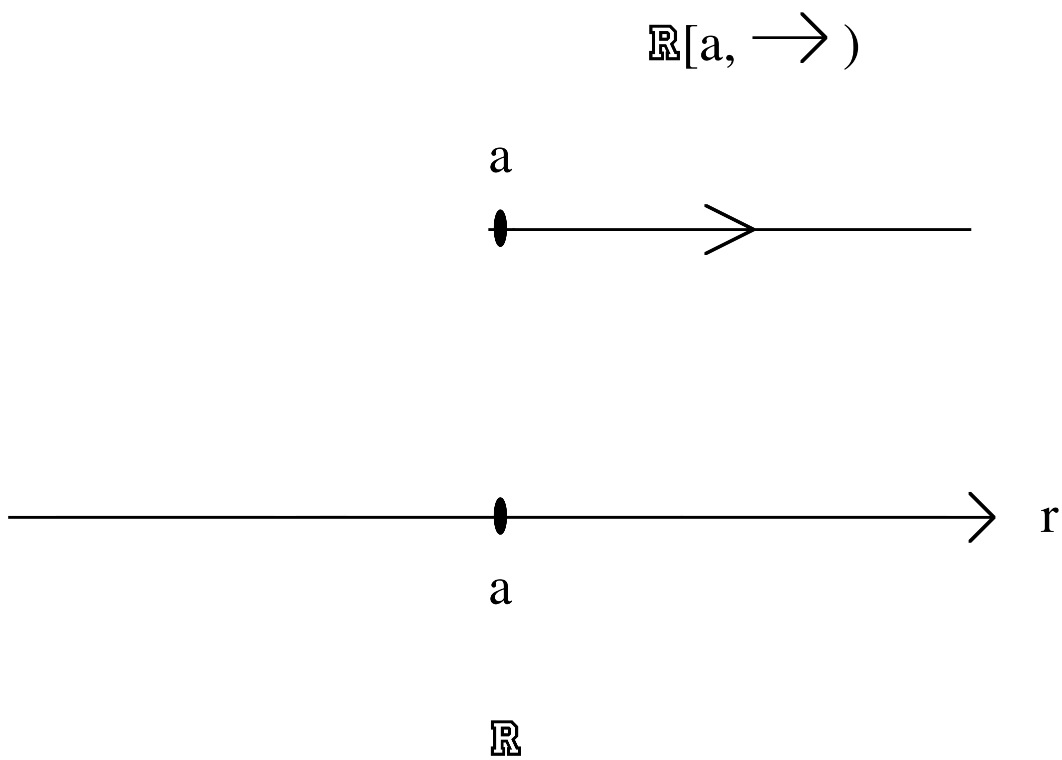
$= \{r \mid r \in \mathbb{R} \ \& \ a \leq r\}$

$= \mathbb{R}[a, \ \rightarrow)$

&

$r \in \mathbb{R}[a, \rightarrow) \ \Leftrightarrow \ r \in \mathbb{R} \ \& \ a \leq r$

pictured horizontally



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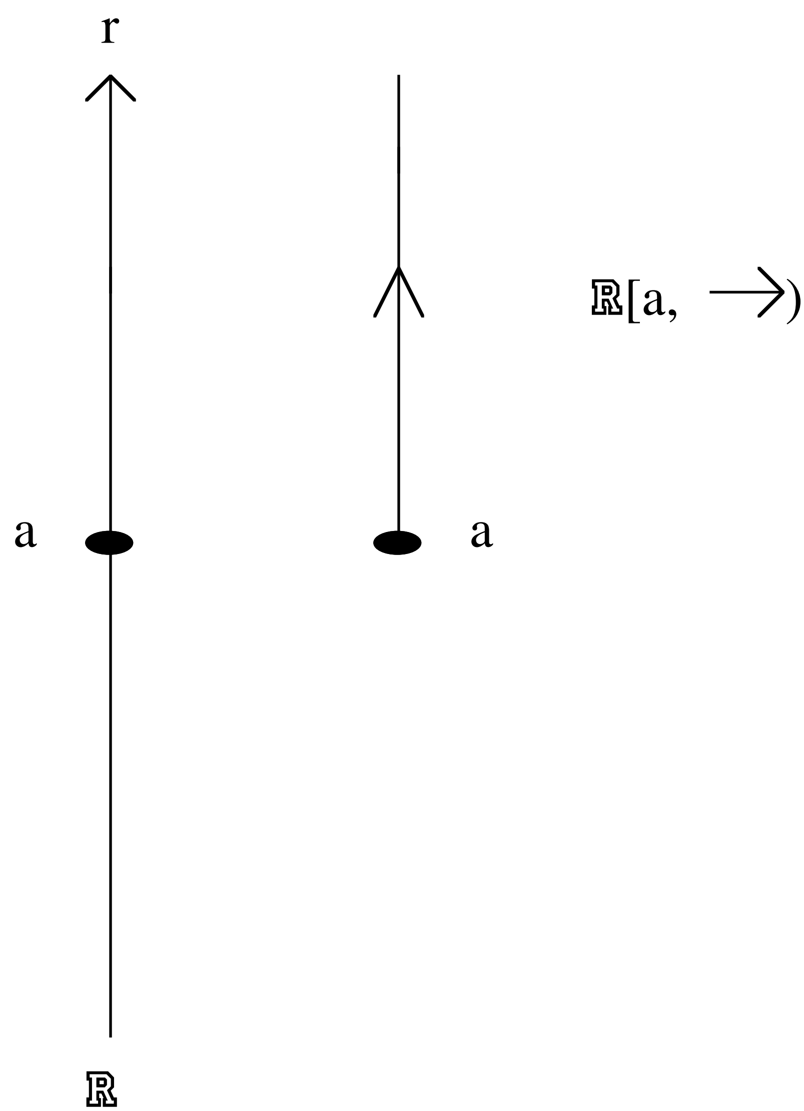


GG34-82

• the open right ray of $\mathbb{R}$

from a

wh $a \in \mathbb{R}$

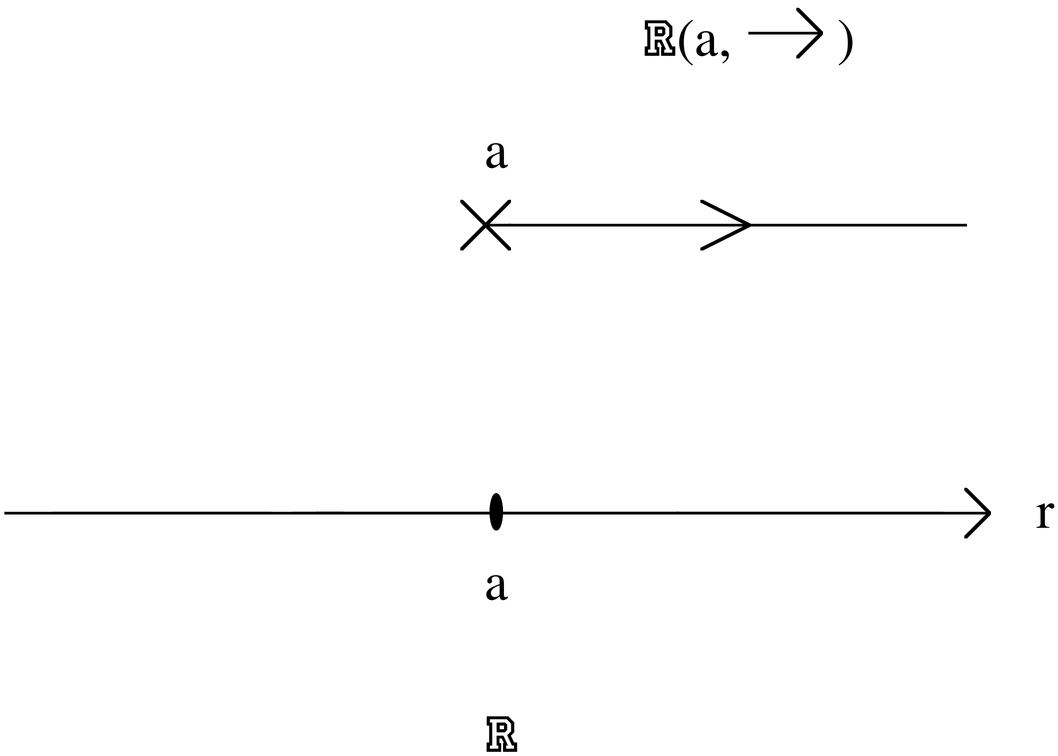
$= \{r \mid r \in \mathbb{R} \ \& \ a<r\}$

$= \mathbb{R}(a, \ \rightarrow)$

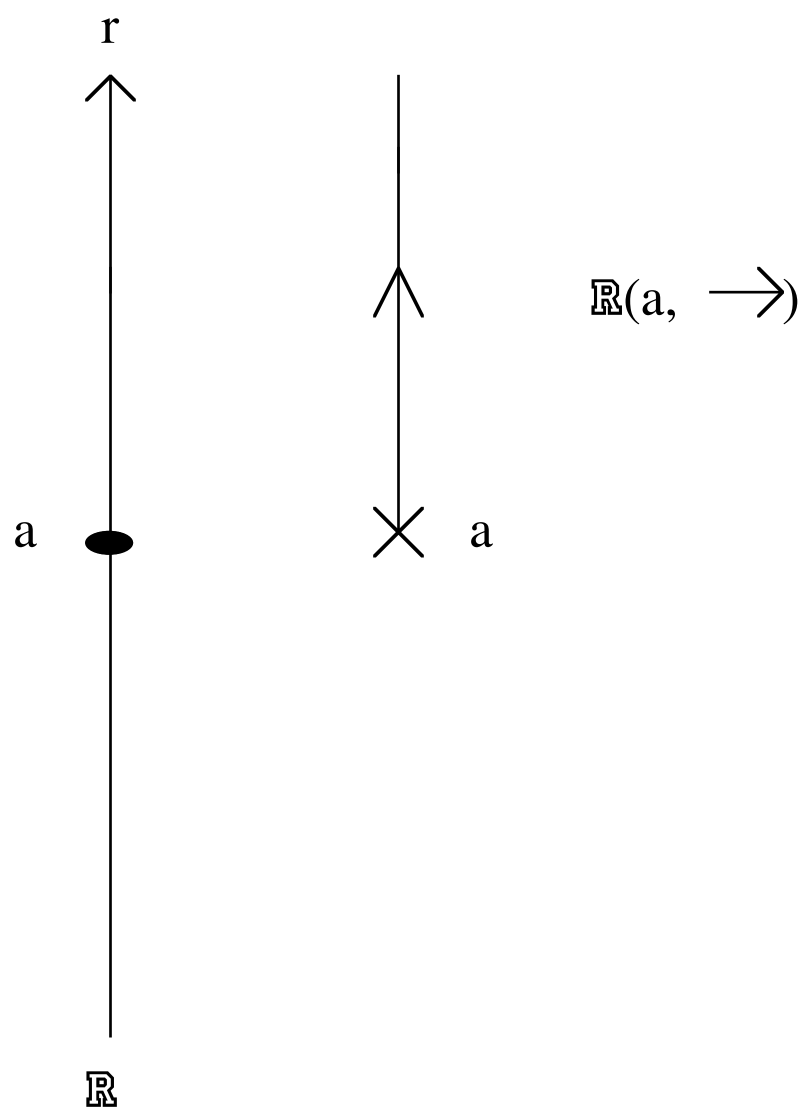
&

$r \in \mathbb{R}(a,\rightarrow) \ \Leftrightarrow \ r \in \mathbb{R} \ \& \ a < r$

pictured horizontally



pictured vertically



GG34-85

• the closed left ray of $\mathbb{R}$

from a

wh $a \in \mathbb{R}$

$= \{r \mid r \in \mathbb{R} \ \& \ r \leq a\}$

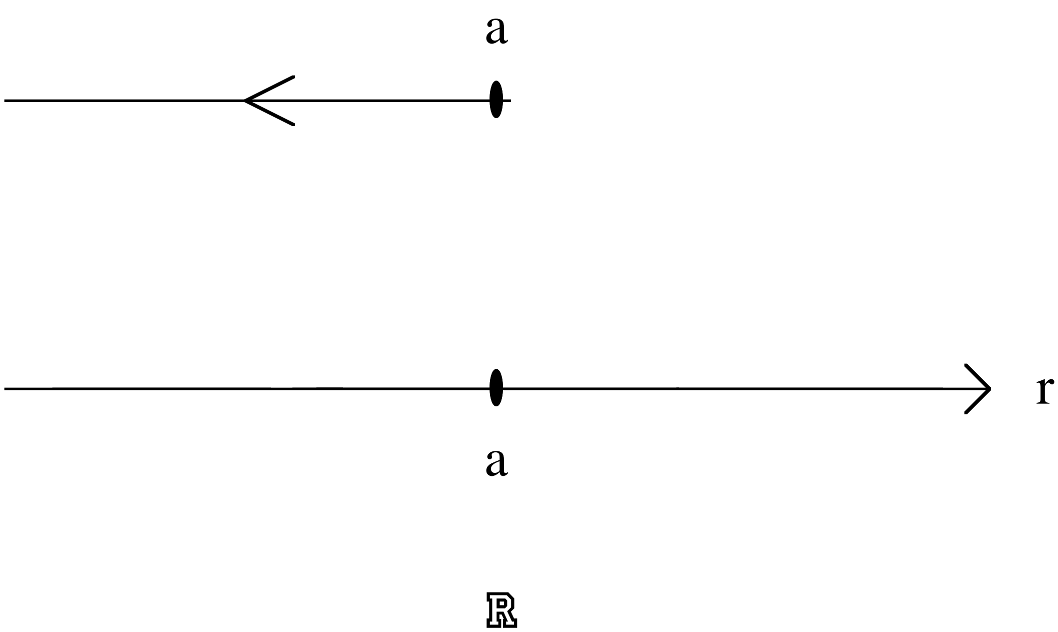
$= \mathbb{R}(\leftarrow, a]$

&

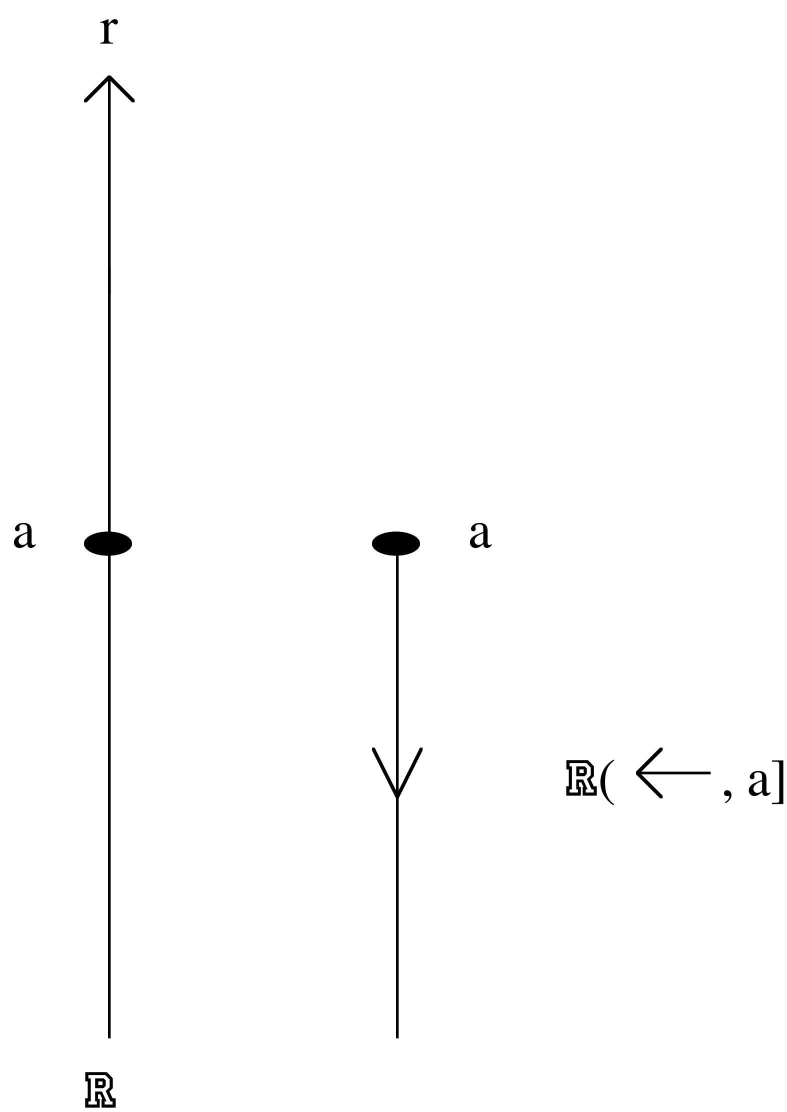
$r \in \mathbb{R}(\leftarrow, a] \iff r \in \mathbb{R} \ \& \ r \leq a$

pictured horizontally

$\mathbb{R}(\longleftarrow, a]$



pictured vertically



GG34-88

• the open left ray of $\mathbb{R}$

from a

wh $a \in \mathbb{R}$

$= \{r \mid r \in \mathbb{R} \ \& \ r<a\}$

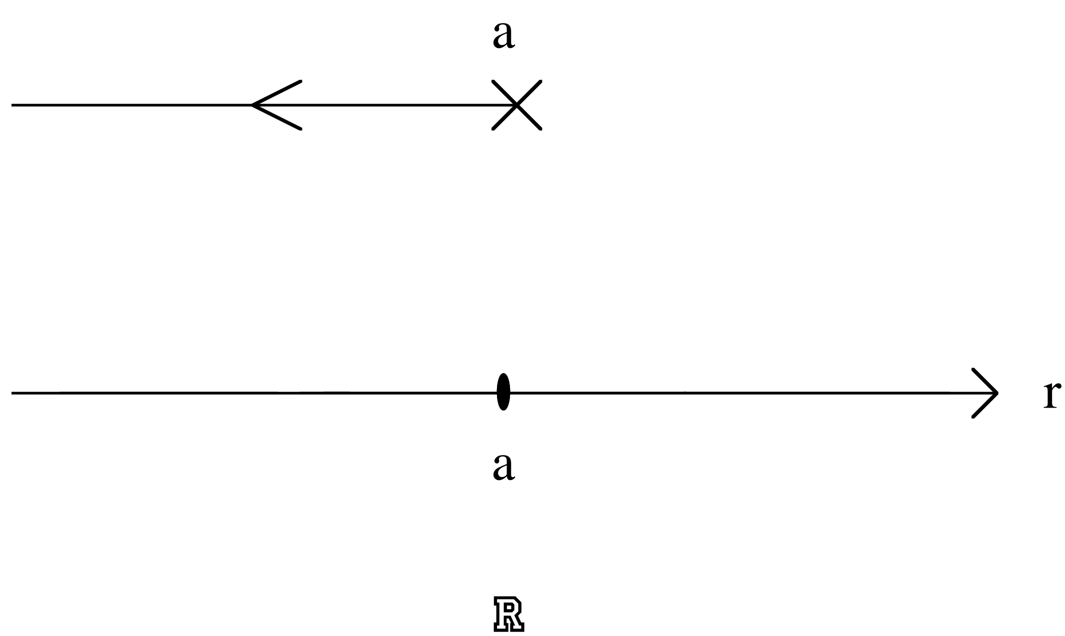
$= \mathbb{R}(\leftarrow, a)$

&

$r \in \mathbb{R}(\leftarrow, a) \iff r \in \mathbb{R} \ \& \ r < a$

pictured horizontally

$\mathbb{R}(\leftarrow, a)$



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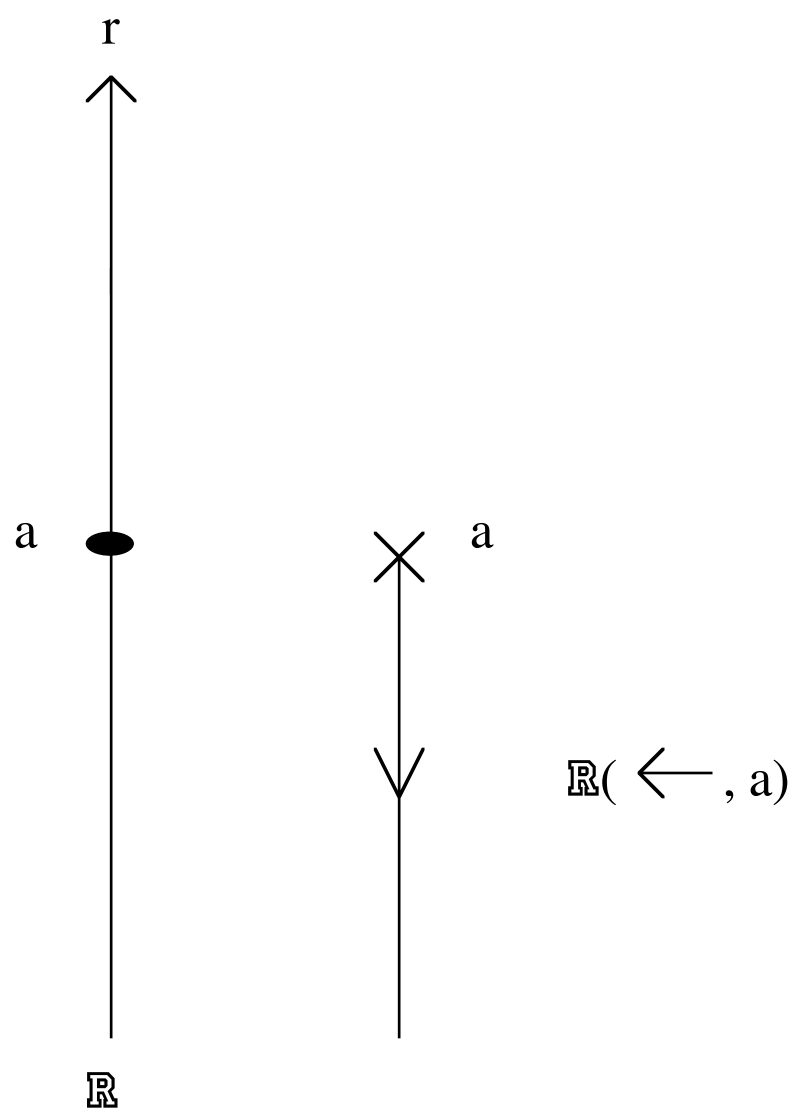


☐ order-theoretic notions corresponding to

- toset ....
- element of toset ....
- element a is strictly less than element b.........................
- element a is strictly greater than element b..................
- intervals ....
- rays ....
- endpoints ....
- inpoints ....
- expoints ....
- boundary ....

□ geometric/topological notions

- (subset of) straight line
- point of line
- point a is to the left of point b; point a is below point b
- point a is to the right of point b; point a is above point b
- segments
- half-lines
- boundary points
- interior points
- exterior points
- boundary

N. the following order notions for posets
are defined in order-dual pairs

two order-dual statements
are numbered
(n) and (n')
where n is a postive integer

a self-dual statement
is numbered
(n = n')
where n is a postive integer

N. it is often the case
that a symbol for a math object
is made up by choosing
all or some of the initial letters
of the words in its name

D. bounds

let

• $X \in$ poset & $E \subset X$

then

(1) a lower bound of E in X

$=_{df}$ an element a of X st

$a \leq x \quad (\forall x \in E)$

(1') an upper bound of E in X

$=_{df}$ an element a of X st

$a \geq x \quad (\forall x \in E)$

(2) the lower - bound set of E in X

$=_{dn}$ lbs E wh lbs $=_{rd}$ lobs

$=_{df}$ the set of all lower bounds of E in X

(2') the upper - bound set of E in X

$=_{dn}$ ubs E wh ubs $=_{rd}$ ubs

$=_{df}$ the set of all upper bounds of E in X

(3) E is bounded (from) below in X

= E is bounded on the left in X

= E is left bounded in X

$=_{df}$ there exists a lower bound of E in X

$\Leftrightarrow$ E has a lower bound in X

$\Leftrightarrow$ $\text{lbs}\, E \neq \varnothing$

(3′) E is bounded (from) above in X

= E is bounded on the right in X

= E is right bounded in X

$=_{df}$ there exists an upper bound of E in X

$\Leftrightarrow$ E has an upper bound in X

$\Leftrightarrow$ $\text{ubs}\, E \neq \varnothing$

(4 = 4') E is unilaterally bounded in X

$=_{df}$ E is bounded below in X

or

E is bounded above in X

(5 = 5') E is (bilaterally) bounded in X

$=_{df}$ E is bounded below in X

and

E is bounded above in X

(6) E is unbounded (from) below in X

= E is unbounded on the left in X

= E is left unbounded in X

$=_{df}$ E is not bounded below in X

(6') E is unbounded (from) above in X

= E is unbounded on the right in X

= E is right unbounded in X

$=_{df}$ E is not bounded above in X

(7 = 7') E is unbounded in X

$=_{df}$ E is unbounded below in X

or

E is unbounded above in X

perhaps both

(8 = 8') E is unilaterally unbounded in X

$=_{df}$ E is unbounded below in X

or

E is unbounded above in X

but not both

(9 = 9') E is bilaterally unbounded in X

$=_{df}$ E is unbounded below in X

and

E is unbounded above in X

D. extremes

let

• $X \in$ poset & $E \subset X$

then

(1) the least element of E

$=_{dn}$ lst E wh lst $=_{rd}$ least

$=_{df}$ the unique element a of E st

$a \leq x \quad (\forall x \in E)$ iie

= the unique lower bound of E that belongs to E iie

(1') the greatest element of E

$=_{dn}$ grt E wh grt $=_{rd}$ greatest

$=_{df}$ the unique element a of E st

$a \geq x \quad (\forall x \in E)$ iie

= the unique upper bound of E that belongs to E iie

(2 = 2') an extreme (element) of E

$=_{df}$ lst E or grt E iie

(3) the greatest lower bound of E in X

$=_{dn}$ glb E wh glb $=_{rd}$ glob

= the infimum of E in X

$=_{dn}$ inf E wh inf $=_{rd}$ inf

= the meet of E in X

$=_{dn}$ Λ E wh Λ $=_{rd}$ meet

$=_{df}$ grt lbs E iie

= the greatest of the lower bounds of E iie

(3') the least upper bound of E in X

$=_{dn}$ lub E wh lub $=_{rd}$ lub

= the supremum of E in X

$=_{dn}$ sup E wh sup $=_{rd}$ soop

= the join of E in X

$=_{dn}$ V E wh V $=_{rd}$ join

$=_{df}$ lst ubs E iie

= the least of the upper bounds of E iie

R. characterization of

least & greatest

let

• $X \in$ poset & $a \in X$ & $E \subset X$

then

(1) $\exists \text{lst}\, E = a$

$\Leftrightarrow$

$a \in E$ & $\forall x \in E . a \leq x$

(1') $\exists \text{grt}\, E = a$

$\Leftrightarrow$

$a \in E$ & $\forall x \in E . a \geq x$

R. characterization of

greatest lower bound & least upper bound

let

- $X \in \text{poset}$ & $a \in X$ & $E \subset X$

then

(1) $\exists \,\text{glb}\, E = a$

$\Leftrightarrow$

$\forall x \in X . a \geq x \Leftrightarrow (\forall y \in E . x \leq y)$

(1') $\exists \,\text{lub}\, E = a$

$\Leftrightarrow$

$\forall x \in X . a \leq x \Leftrightarrow (\forall y \in E . x \geq y)$

D. extrema

let

• $X \in$ poset & $E \subset X$

then

(1) a minimal element of E

= a minimum (element) of E

$=_{df}$ an element a of E st

$\neg \exists\, x \in E\,.\, x < a$ iie

ie no element of E is strictly less than a iie

(1') a maximal element of E

= a maximum (element) of E

$=_{df}$ an element a of E st

$\neg \exists\, x \in E\,.\, x > a$ iie

ie no element of E is strictly greater than a iie

(2 = 2') an extremal element of E

= an extremum (element) of E

$=_{df}$ a minimal element of E

or

a maximal element of E

D. extrema for tosets

let

• X ∈ toset

• E ∈ nonempty finite subset of X

then

(1) the minimal element of E

= the minimum (element) of E

$=_{dn}$ min E wh min $=_{rd}$ min

$=_{df}$ lst E

which necessarily exists uniquely

(1') the maximal element of E

= the maximum (element) of E

$=_{dn}$ max E wh max $=_{rd}$ max

$=_{df}$ grt E

which necessarily exists uniquely

N. notation for the order types of some tosets;
lowercase Greek letters are traditionally used
ort $=_{df}$ order type (of)

$\omega = \text{ort}\,\mathbb{N} = \text{ort}\,\mathbb{P}$

$^*\omega = \text{ort}\,(-\mathbb{N}) = \text{ort}\,(-\mathbb{P})$

$\pi = \text{ort}\,\mathbb{Z}$

$\eta = \text{ort}\,\mathbb{Q} = \text{ort}\,\mathbb{Q}(0,1)$

$\vartheta = \text{ort}\,\mathbb{I} = \text{ort}\,\mathbb{R}[0,1]$

$\lambda = \text{ort}\,\overset{o}{\mathbb{I}} = \text{ort}\,\mathbb{R}(0,1) = \text{ort}\,\mathbb{R}$

also positive integers for nonempty finite tosets;
addition & multiplication of order types are definable;
a calculus of order types may be developed

D. well ordered sets

(1) a well ordered set

$=_{ab}$ woset

$=_{df}$ a poset or toset X st

every nonempty subset of X

has a least element

= a toset st

$\leq / <$ is left - founded

= a toset st

$\geq / >$ is right - founded

(1') an inversely well ordered set

$=_{df}$ iwoset

$=_{df}$ a poset or toset X st

every nonempty subset of X

has a greatest element

= a toset st

$\leq / <$ is right - founded

= a toset st

$\geq / >$ is left - founded

(2) a well ordering of a set X

$=_{df}$ a partial or total order of X

that makes X a woset

= a left - founded total order of X

(2') an inverse well ordering of a set X

$=_{df}$ a partial or total order of X

that makes X an iwoset

= a right - founded total order of X

D. ordinal numbers

• the historical definition:
an ordinal number
$=_{df}$ the order type of a well ordered set

• a more modern definition:
an ordinal number
$=_{df}$ a woset X st
$a \in X \Rightarrow X(\leftarrow, a) = a$;
the well ordering is
the elementhood relation $\in$
& also equivalently
the subset relation $\subset$

• ordinal number
$=_{ab}$ ordinal

C. the von Neumann construction of ordinals

- an ordinal = the set of all smaller ordinals

where the order relation is
the elementhood relation $\in$
or equivalently
the subset relation $\subset$

- thus

$0 = \varnothing$
$1 = \{0\}$
$2 = \{0,1\}$
$3 = \{0,1,2\}$
etc
$\omega = \{0,1,2,\cdots\}$
$\omega+1 = \{0,1,2,\cdots,\omega\}$
$\omega+2 = \{0,1,2,\cdots,\omega,\omega+1\}$
$\omega+3 = \{0,1,2,\cdots,\omega,\omega+1,\omega+2\}$
etc

D. cardinal numbers

• Ord $=_{df}$ the class of all ordinals
well ordered by $\in$ and by $\subset$

• $\alpha, \beta \in$ var Ord

• let $\sim$ denote
the reflexive symmetric transitive relation
= the equivalence relation
in Ord
st
$\alpha \sim \beta$ iff there exists a one - to - one map of α onto β

• a cardinal number
$=_{df}$ the least ordinal
in an equivalence class of Ord modulo $\sim$

• cardinal number
$=_{ab}$ cardinal

☐ tabular form for the four inequalities

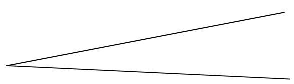
inequality → ↓	lohi	hilo
weak	≤	≥
strict	<	>

☐ tabular form for the four negated inequalities

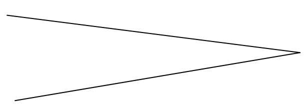
negated inequality → ↓	lohi	hilo
weak	≰	≱
strict	≮	≯

□ mnemonics

• the strict inequality signs
<
and
>
are suggestive of the musical signs



and



for crescendo and diminuendo;
thus
soft < loud
and
loud > soft
thinking of
bigger numbers as noisier numbers
&
smaller numbers as quieter numbers

• think of >
as a reducing machine
which shrinks a larger number on the left
to a smaller number on the right;
think of <
as an enlarging machine
which expands a smaller number on the left
to a larger number on the right

☐ the less/greater terminology
has many variants
eg

- below/above
- inferior/superior
- infra/supra
- left/right
- littler/bigger
- lower/higher
- lower/upper
- minorant/majorant
- minorize/majorize

- precede/succeed
- predecessor/successor
- smaller/larger
- subordinate/dominant
- subordinate/dominate
- under/over
- underestimate/overestimate

etc