Math Snippets: First Bouquet

#28 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
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☐ four classical geometrical terms of measurement

• rectification
= to make a straight line out of a curve
& thus find its length
('straighten the curve')

• quadrature
= to make a square out of a plane region
& thus find its area
('square the plane region'; recall 'square the circle')

• complanation
= to make a plane region out of a curved surface
& thus find its area
('flatten the surface')

• cubature
= to make a cube out of a solid region
& thus find its volume
('cube the solid')

in more detail

• ‘finding the length of’
has the classical geometrical name of
‘rectification’
which is conceived of as reduction of an arc
to a straight line (= rectilinear) segment of equal length
whose length can then be measured in principle;
‘rectification’
thus means
‘making into a straight line of equal length
& taking its length’

• ‘finding the area of’
for a plane region
has the classical geometrical name of
‘quadrature’
which is conceived of as reduction of a plane region
to a square (= quadrate) of equal area
whose area can then be measured in principle;
‘quadrature’
thus means
‘making into a square of equal area
& taking its area’

• ‘finding the area of’
for a nonplanar region
has the classical geometrical name of
‘complanation’
which is conceived of as reduction of a nonplanar region
to a plane region of equal area
whose area can then be measured in principle;
‘complanation’
thus means
‘making into a plane region of equal area
& taking its area’

• ‘finding the volume of’
has the classical geometrical name of
‘cubature’
which is conceived of as reduction of a sold region
to a cube of equal volume
whose volume can then be measured in principle;
‘cubature’
thus means
‘making into a cube of equal volume
& taking its volume’

• presumably of
the one-dimensional, two-dimensional, three-dimensional figures,
the straight line segment, the square, the cube
are the simplest figures
& the easiest to measure in principle;
therefore they are to be taken as
the standards of measure
of length, area, volume;
in particular
the units of length, area, volume
are the length, area, volume
of the unit segment, unit square, unit cube

☐ the 7C's of good mathematical exposition

good math exposition should be the 7C's:

- correct
- clear
- clean
- crisp
- concise
- comprehensive
- congenial

□ graded adjectives of return
as applied to a point

- fixed
- periodic
- regularly almost periodic
- almost periodic
- recurrent
- fleeing
- wandering

□ notational/terminological compression

• the set of all items
= the set of items
= the item set
= the items

if the set of all items is also provided with a structure
so that it becomes a system viz
system = (set, structure)
or paraphrasing
system = set + structure (the triple ess dictum)
then
• the system of all items
= the system of items
= the item system

the same symbol may be used
for both ‘set’ & ‘system’
eg
• the set of all integers
= the set of integers
= the integer set
= the integers
= $\mathbb{Z}$
• the ring of all integers
= the ring of integers
= the integer ring
= $\mathbb{Z}$

☐ alias & alibi

• alias & alibi are two interesting loan words from Latin
with the same first three letters and
with fourth letters the first two letters of the alphabet

• alias = at another time (in Latin)
&
otherwise called (in English)

• alibi = at another place (in Latin)
&
plea of innocence because elsewhere (in English)

• alias & alibi
have suggestive uses in geometry inp vector spaces
viz
an alias is a coordinate change of points
&
an alibi is a transformation of points

• the same equations may have two interpretations
viz
the alias interpretation
as defining a coordinate change of points
&
the alibi interpretation
as defining a transformation of points

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☐ atlas & maps

• to define an 'atlas' as a set of maps
makes sense from both
a geographical & a mathematical POV

• Gerhardus Mercator
= Latinized form of
Gerhard Kremer
1512 - 1594
Flemish
cartographer, geographer

introduced the word 'atlas'
in its cartographic/geographic sense
when the figure of Atlas,
the ancient Greek mythological Titan
supporting the world,
was used to decorate the title page
of Mercator's map collection of 1585;
in 1569 to aid navigation
he introduced the Mercator map projection
in which the path of a ship steering on a constant bearing
is represented by a straight line on the map;
the Mercator map projection has been used
for nautical charts ever since

• the above Mercator of Mercator maps
is not the same Mercator
as the Mercator of Mercator's series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (-1 < x \leq 1)$$

• the series Mercator was

Nicolaus Mercator (né Kaufmann)
ca 1620 - 1687
German, born in Denmark, lived in England
mathematician, astronomer, engineer;
engineer in the construction
of the fountains of Versailles

• note that
'mercator'
is a Latin word meaning
'merchant, buyer'
which is also the meaning
of the above two original family names

D. the binomial coefficients $\binom{n}{r}$

are definable for any integers n & r

as follows:

- $\binom{n}{r} = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$ if $r > 0$

- $\binom{n}{r} = 1$ if $r = 0$

- $\binom{n}{r} = 0$ if $r < 0$

☐ on circles / spheres & rates of change

Δ on circles

let

• $C =_{df}$ the circumference of a circle with radius r

• $A =_{df}$ the area of a circle with radius r

then

• it is geometrically clear that

$$\Delta A \approx C \Delta r$$

$$\frac{\Delta A}{\Delta r} \approx C$$

$$\frac{dA}{dr} = C \quad \text{taking the limit as } \Delta r \to 0$$

ie

the rate of change

of the area of a circle wrt the radius

equals

the circumference

• in formulas

$$A = \pi r^2$$

$$C = 2 \pi r$$

$$\frac{dA}{dr} = C$$

Δ on spheres

let

• $S =_{df}$ the surface area of a sphere with radius r

• $V =_{df}$ the volume of a sphere with radius r

then

• it is geometrically clear that

$$\Delta V \approx S \Delta r$$

$$\frac{\Delta V}{\Delta r} \approx S$$

$$\frac{dV}{dr} = S \quad \text{taking the limit as } \Delta r \to 0$$

ie

the rate of change

of the volume of a sphere wrt the radius

equals

the surface area

• in formulas

$$V = \frac{4}{3} \pi r^3$$

$$S = 4 \pi r^2$$

$$\frac{dV}{dr} = S$$

□ classes of sets
that are not sets

• the class of all sets
= the class of sets
= the set class
= Set

• the class of all ordinals
= the class of ordinals
= the ordinal class
= Ord

• the class of all cardinals
= the class of cardinals
= the cardinal class
= Crd

• Set $\supset$ Ord $\supset$ Crd
no one of which is a set

• instead of saying
set_0 , set_1 , set_2 , $\cdots$
say
set, class, collection, $\cdots$;
everything is still a ' set' ;
the distinction among
set, class, collection, $\cdots$
is just a notational / terminological device
for the sake of clarity & simplicity

• further examples of classes that are not sets:
(1) the class of all partially ordered sets = Pos
(2) the class of all groups = Grp
(3) the class of all topological spaces = Top

□ unique existence theorem
for general determinants

• the determinant function
on the set of all $n \times n$ matrices
(where n is a positive integer)
over a given field
is uniquely determined by
the following five (equivalent & dual) conditions:

(1) the determinant is field-valued

(2) the determinant is linear in each row (column)

(3) the determinant is skew-symmetric in rows (columns)

(4) the determinant is invariant
under the addition to any row (column)
of any field multiple of any other row (column)

(5) the determinant of the identity matrix is unity

☐ don't knock memory

• from Pensées (French) (=Thoughts) by Pascal

La mémoire est nécessaire pour toutes les opérations
de la raison.
= Memory is necessary for all the workings of reason.

• bioline
Blaise Pascal
1623-1662
French
combinatorist, geometer, probabilist, physicist;
founder of theory of probability together with Fermat;
essayist, philosopher;
inventor of
the bus & the wheelbarrow & a calculating machine;
abandoned mathematics & science
for metaphysics & theology
and because of this, has been called
'without doubt the greatest might-have-been
in the history of mathematics';
(note: the word 'bus'
comes from the Latin dative case 'omnibus'
meaning 'for all')

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□ the strategy of divide & conquer

• there is a general strategy of solving/proving
a mathematics problem/theorem
whether it be at the level of pioneering research
or at the level of a textbook exercise;
the strategy is
to divide/separate the problem/theorem
into smaller & presumably easier parts
and then to solve/prove
these lesser/easier problems/theorems
thereby solving/proving the original problem/theorem;
this strategy is called
the strategy of divide & conquer;
the great difficulty in applying this strategy
is to see how to 'divide'

• there is a related ancient Roman political maxim

divide et impera (Latin)
= DEE-wih-deh et IHM-peh-rah
= divide and rule

which was so successful that it was adopted by Machiavelli

• bioline
Niccolò Machiavelli
1469-1527
Italian
statesman, writer, political philosopher;
his famous masterpiece book 'The Prince'
was written in 1513
and published in 1532;
it purports to be a handbook for rulers
in which the guiding principle is that
the end justifies the means;
his name has contributed a word to the English language:
Machiavellian

□ the euclidean algorithm

Δ the euclidean algorithm
is the following mechanical procedure
for finding
the greatest common divisor of a and b
= GCD(a, b) = gcd(a, b)
aka
the highest common factor of a and b
= HCF(a, b) = hcf(a, b)
where a and b are
two unequal positive integers
with $a > b$:
divide the larger dividend a
by the smaller divisor b
to obtain
the positive integer quotient q
&
the nonnegative integer remainder r
which is less than the divisor b;
thus
$a = bq + r,\ 0 \leq r < b$;
repeat the process of division,
now dividing b by r if r is positive;
continue the procedure so that
the strictly decreasing sequence
of remainders must eventually reach zero;
the last nonzero remainder
is the gcd(a, b)

Δ a convenient pattern to use repeatedly
in recording the results of the calculations
is this

• dividend
↓
• divisor → quotient
↓
• remainder

which may be read

dividend *
by divisor *
gives
quotient *
with remainder *
where the stars represent numerals

eg

find gcd (6097, 1139)

- 6097

  $\downarrow$

- 1139 $\rightarrow$ 5

  $\downarrow$

- 402 $\rightarrow$ 2

  $\downarrow$

- 335 $\rightarrow$ 1

  $\downarrow$

- 67 $\rightarrow$ 5

  $\downarrow$

- 0

$\therefore$ gcd (6097, 1139) = 67

☐ the standard universal symbol for the empty set

• the character
Ø
read ‘oh slash’ or ‘slash oh’
is now the standard universally used and recognized symbol
for the empty/null/vacuous/void set;
it may also be read simply ‘empty’;
the predicate x $\notin$ Ø is always true
&
the predicate x $\in$ Ø is always false;
the symbol Ø for the empty set
was first used by
André Weil (1906-1998, French),
a founding member of Bourbaki,
in the 1930’s;
Ø ø is a vowel letter
in the Danish, Faroese, and Norwegian alphabets
and it is also used as a phonetic symbol
in the International Phonetic Alphabet;
it has the phonetic value of the vowel sound
in the French word
feu = fire
&
the German word
schön = beautiful;
the sound Ø does not occur in English

☐ some fundamental theorems by name

• the fundamental theorem
of (the subject)/(the objects)

• subjects
algebra
arithmetic
calculus
Galois theory
information theory
projective geometry

• objects
closed topological surfaces
differentiable curves
differentiable surfaces
local Lie groups
symmetric polynomials
topological curves
ultraproducts

☐ verbal paraphrases
of the equality of a and b viz
a = b

- a equals b
- a is equal to b
- a and b are equal
- a and b are equal to each other
- a is identical to b
- a is identical with b
- a and b are identical
- a and b are identical to each other
- a and b are identical with each other
- a coincides with b
- a and b coincide
- a and b coincide with each other
- a and b are coincident
- a and b are coincident with each other
- a is the same as b
- a is the same thing as b
- a and b are the same
- a and b are the same thing
- a and b are one
- a and b are one thing
- a and b are one and the same
- a and b are one and the same thing
- a is indistinguishable from b
- a and b are indistinguishable
- a and b are indistinguishable from each other

pairwise equivalent symbolic statements

- $a = b$
- $\{a\} = \{b\}$
- $\{a, b\} = \{a\}$
- $\{a, b\} = \{b\}$
- $(a, b) = (b, a)$
- $\mathrm{crd}\{a, b\} = 1$
- $\forall f \,.\, fa \Rightarrow fb$
- $\forall f \,.\, fa \Leftarrow fb$
- $\forall f \,.\, fa \Leftrightarrow fb$

the last three conditions are from
the second - order predicate calculus
= second - order logic
in which equality of individuals
is definable in this way

□ math vs mind

• it seems that
mathematics comes from
the working of the mind

• ¿ is mathematics how the mind works ?

• ¿ is mathematics a reflection
of how the mind works ?

• ¡ Shades of Platonic Ideal Forms !

• ¿ is it possible
to describe within mathematics
how the mind works ?

• ¿ can the mind understand the brain ?

• ¿ can the brain/mind understand itself ?

• ¿ is there some mathematical theorem
(say like Gödel's theorems)
which says that
the brain/mind cannot understand itself ?

☐ some random questions

• ¿ to what extent is physical intuition
a reliable guide in geometry ?
evidently Riemann relied heavily on it

• ¿ does the notion of a vector
come from a pointing finger ?

• ¿ what is the relationship between
mental imagery
&
visual perception ?

• ¿ what theorems have nicknames ?

• ¿ how successfully can
Braille
&
Ameslan = American Sign Language
express mathematics ?
¿ can
Braille's symbols of raised dots on paper
&
Ameslan's hand/body configurations/motions
be employed to any mathematical advantage ?