Angles on Angles

#16 of Gottschalk's Gestalts

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of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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500 Angell St #414
Providence RI 02906

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☐ the notion of plane angle

Δ we recognize three kinds of plane angles:
sectorial angles
rotary angles
radial angles

• plane angle as infinite circular sector
= plane angle as sector
= plane angle as sectorial angle
= sectorial plane angle
= sectorial angle

• plane angle as rotation of a ray about its vertex
= plane angle as rotation
= plane angle as rotary angle
= rotary plane angle
= rotary angle

• plane angle as union of rays in the plane
with a common vertex
= plane angle as ray-union
= plane angle as radial angle
= radial plane angle
= radial angle

∆ the notion of radial angle

generalizes the notion of sectorial angle;
a sectorial angle is a special kind of radial angle;
radial angles that are not sectorial angles
are not ordinarily encountered until after
calculus level math courses;
but it is the notion of radial angle
that extends easily to higher dimensions
and in particular to solid (= three-dimensional) angles

∆ as with all geometric objects

a basic question arises as to
how to measure the magnitude
of these various kinds of angle

□ pictures of sectorial & rotary angles

• to draw a picture of a sectorial angle
draw the sides from the vertex
and draw a circle arc centered at the vertex
that lies between the sides
and within the angle;
to indicate another sectorial angle
on the same diagram
draw two such circle arcs close together
or mark a circle arc
with a short tick (crosspiece)
in the middle of the arc;
increase numbers of arcs and of ticks on arcs
to indicate more sectorial angles

• to identify a right sectorial angle by sight
complete a small square in the corner

• to draw a picture of a rotary angle
draw the sides from the vertex
and draw an expanding/contracting
counterclockwise/clockwise pointed spiraling arc
from the initial side to the terminal side
to indicate the direction and the numerical value
of the rotation

□ visualizing a radial angle

• here is a way of thinking of a radial angle;
a radial angle is determined by
a point in the plane,
called the vertex V,
and
any nonempty set in the plane,
called the directrix D;
take the union A of all rays (= half-lines) from V
passing thru a point in D;
the set A is a radial angle;
it is said that
the directrix D subtends the angle A
from the vertex V
and
the angle A is subtended by the directrix D
from the vertex V;
if D lies on the unit circle centered at V,
then the numerical 'length' of D
is taken to be the measure in radians
of the radial angle A

□ the notion of spacial angle

Δ we recognize two kinds of spacial angles:
dihedral angles
solid angles

• a dihedral angle,
visualized as two pages in a book
with both pages stationary
or one page stationary and the other turning,
can be thought of
as extending the two notions of
sectorial plane angle
and
rotary plane angle
to spacial angles;
a perpendicular plane section of a dihedral angle
produces
one of these two kinds of plane angle
whose measure is taken to be
the measure of the dihedral angle

• a solid angle
is defined to be
a union of rays in space
with a common vertex;
a solid angle is a ray-union;
a solid angle is a kind of radial (= made up of rays) angle
and indeed extends the notion of radial plane angle
to spacial angles

• a sectorial dihedral angle
is a special kind of a solid angle
but the measure
of the plane section of the dihedral angle
differs from the measure as solid angle

☐ visualizing a solid angle

• here is a way of thinking of a solid angle;
a solid angle is determined by
a point in space,
called the vertex V,
and
any nonempty set in space,
called the directrix D;
take the union A of all rays (=half-lines) from V
passing thru a point in D;
the set A is a solid angle;
it is said that
the directrix D subtends the angle A
from the vertex V
and
the angle A is subtended by the directrix D
from the vertex V;
if D lies on the unit sphere centered at V,
then the numerical 'area' of D
is taken to be the measure in steradians
of the solid angle A

□ units of angle measure

• for sectorial or rotary plane angles
there are two primary units of measure
that concern us
viz
the dimensionless unit of plane angle measure called
degree
(with subunits called 'minute' and 'second')
and
the dimensionless unit of plane angle measure called
radian

• for solid angles
there is one primary unit of measure
that concerns us
viz
the dimensionless unit of solid angle measure called
steradian

• both the radian and the steradian are
supplementary (because dimensionless) SI units
ie
units in the
Système International d'Unités (French)
= International System of Units
which is the standard system of scientific units
used worldwide

□ the measure in degrees = the degree measure of sectorial angles

• by definition

one round angle

= 360 degrees = 360 deg = 360^{deg} = 360^{o}

1 degree = 1^{o} = 60 minutes = 60 min = 60^{min} = 60'

1 minute = 1' = 60 seconds = 60 sec = 60^{sec} = 60''

1 second = 1'' = 1 sec = 1^{sec}

□ a vanishing but useful notation

• given an angle A
then its measure
in whatever unit that would be understood
may be denoted by
meas A or simply mA;
however
if no ambiguity can reasonably arise,
then the measure of A
may be denoted by A itself

□ classification & names of sectorial angles characterized by degree measure

• A is a sectorial angle:

$0^o \leq A \leq 360^o$

• A is a zero angle:

$A = 0^o$

• A is an acute angle:

$0^o < A < 90^o$

• A is a right angle:

$A = 90^o$

• A is an obtuse angle:

$90^o < A < 180^o$

• A is a straight angle:

$A = 180^o$

• A is a reflex - obtuse angle:

$180^o < A < 270^o$

• A is a reflex - right angle:

$A = 270^o$

• A is a reflex - acute angle:

$270^o < A < 360^o$

• A is a round angle:

$A = 360^o$

note: once around a point
gives a round angle

• A is an oblique angle:

A is an acute angle

or

A is an obtuse angle

• A is a reflex - oblique angle:

A is a reflex - obtuse angle

or

A is a reflex - acute angle

• A is a flex angle:

$0^{o} \leq A \leq 180^{o}$

• A is a reflex angle:

$180^{o} \leq A \leq 360^{o}$

- A is a semisextile angle:

$A = 30^o$

- A is a semiright angle:

$A = 45^o$

- A is a sextile angle:

$A = 60^o$

- A is a trine angle:

$A = 120^o$

- A is a sesquiright angle:

$A = 135^o$

- A is a sesquicental angle:

$A = 150^o$

- A is an n degree angle wh $n \in \mathbb{R}[0,\ 360]$:

$A = n^o$

• A is a quadrant one angle = A is a Q I angle
= A is in quadrant one = $A \in Q\,I$:

$0^o \le A \le 90^o$

• A is a quadrant two angle = A is a Q II angle
= A is in quadrant two = $A \in Q\,II$:

$90^o \le A \le 180^o$

• A is a quadrant three angle = A is a Q III angle
= A is in quadrant three = $A \in Q\,III$:

$180^o \le A \le 270^o$

• A is a quadrant four angle = A is a Q IV angle
= A is in quadrant four = $A \in Q\,IV$:

$270^o \le A \le 360^o$

• A is a quadrantal angle = A is a Q angle
= A is quadrantal = $A \in Q$:

$A = 0^o$ or 90^o or 180^o or 270^o or 360^o

• A is an open quadrant N angle
and
A is in open quadrant N
provided that the corresponding strict inequalities hold,
and a small open circle may be placed above Q as $\overset{o}{Q}$
for the notation (here o stands for ' open');
to emphasize the earlier notion,
say that
A is a closed quadrant N angle
and
A is in closed quadrant N,
and place a horizontal bar above Q as $\overline{Q}$
for the notation

- A and B are complementary angles

= A and B are complementary

= A and B are complements (of each other)

= A is complementary to B

= A is the complement of B :

$A + B = 90^{o}$

- A and B are supplementary angles

= A and B are supplementary

= A and B are supplements (of each other)

= A is supplementary to B

= A is the supplement of B :

$A + B = 180^{o}$

- A and B are replementary angles

= A and B are replementary

= A and B are replements (of each other)

= A is replementary to B

= A is the replement of B :

$$A + B = 270^{o}$$

- A and B are explementary angles

= A and B are explementary

= A and B are explements (of each other)

= A is explementary to B

= A is the explement of B :

$$A + B = 360^{o}$$

☐ the measure in radians = the radian measure of sectorial angles

• by definition

one round angle

$= 2\pi$ radians $= 2\pi$ rad $= 2\pi^{rad} = 2\pi^{r}$

1 radian $=$ 1 rad $= 1^{rad} = 1^{r}$

□ an angle on angle measure:
degree measure vs radian measure;
numerical conversion from one measure to the other

• remember

$$360^{o} = \text{one round angle } = 2\pi^{r}$$

• start with

$$360^{o} = 2\pi^{r}$$

• now divide successively by
the 24 = 4 × 3 × 2 factors of

$$360 = 2^{3} \times 3^{2} \times 5 \quad \text{(pf)}$$

viz

1	360
2	180
3	120
4	90
5	72
6	60
8	45
9	40
10	36
12	30
15	24
18	20

to obtain the quotients

$$360^{o} = 2\pi^{r}$$

$$180^{o} = \pi^{r}$$

$$120^{o} = \frac{2\pi^{r}}{3}$$

$$90^{o} = \frac{\pi^{r}}{2}$$

$$72^{o} = \frac{2\pi^{r}}{5}$$

$$60^{o} = \frac{\pi^{r}}{3}$$

$$45^{o} = \frac{\pi^{r}}{4}$$

$$40^{o} = \frac{2\pi^{r}}{9}$$

$$36^{o} = \frac{\pi^{r}}{5}$$

$$30^{o} = \frac{\pi^{r}}{6}$$

$$24^{o} = \frac{2\pi^{r}}{15}$$

$$20^{o} = \frac{\pi^{r}}{9}$$

$$1^{o} = \frac{\pi^{r}}{180}$$

$$2^{o} = \frac{\pi^{r}}{90}$$

$$3^{o} = \frac{\pi^{r}}{60}$$

$$4^{o} = \frac{\pi^{r}}{45}$$

$$5^{o} = \frac{\pi^{r}}{36}$$

$$6^{o} = \frac{\pi^{r}}{30}$$

$$8^{o} = \frac{2\pi^{r}}{45}$$

$$9^{o} = \frac{\pi^{r}}{20}$$

$$10^{o} = \frac{\pi^{r}}{18}$$

$$12^{o} = \frac{\pi^{r}}{15}$$

$$15^{o} = \frac{\pi^{r}}{12}$$

$$18^{o} = \frac{\pi^{r}}{10}$$

• also note

$$1^{o} = \frac{\pi}{180}^{r} \approx 0.01745+^{r}$$

$$1^{r} = \frac{180^{o}}{\pi} \approx 57.29578-^{o} \approx 57^{o}\,17^{/}\,45-^{//}$$

which contains
a remarkable accidental (?) simultaneous occurrence of 1745
in the two approximations

• now 1745 was a very good year:
it was the year the Leyden jar was invented

☐ measure of circular = circle arcs

• by definition
the measure of a circular = circle arc
is
the measure in
degrees/minutes/seconds/radians/whatever
of the central angle, a sectorial angle,
that has its vertex at the center of the arc
and
that subtends the given arc

• the notion of arc measure
is useful in
astronomy
geodesy
geography
meteorology
space navigation
terrestrial navigation

□ clockwise vs counterclockwise

∆ the meaning of the words

• 'clockwise' & 'counterclockwise'
refer physically
to the two opposite directions of rotary motion
in a horizontal or vertical plane before us

• 'clockwise'
means literally
'in the manner of the clock'
ie
in the manner in which clockhands turn;
clockhands are designed to turn the way they do
because
that is the direction of motion
of the shadow of the gnomon on a sundial
in the northern hemisphere
(it's counterclockwise on a sundial
in the southern hemisphere
but their clocks still turn the same as northerners')

• 'counterclockwise'
means literally
'counter/contrary/opposed/opposite to clockwise'
ie
opposite to the manner in which clockhands turn

∆ meaning of positive & negative rotary directions

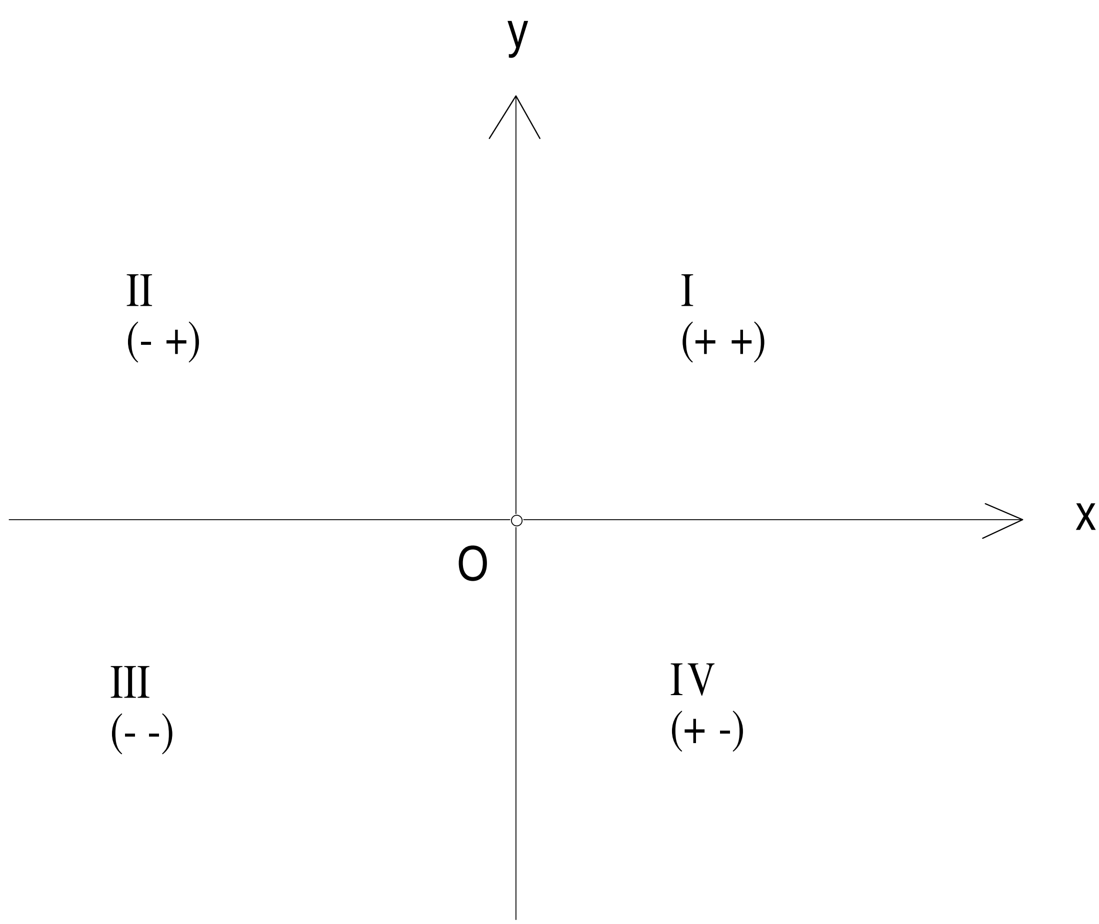
• for a physical plane:

the 'positive' rotary direction for/in/of the physical plane
= df the counterclockwise direction
&
the 'negative' rotary direction for/in/of the physical plane
= df the clockwise direction

• for a geometric plane
provided with
a rectangular (x, y) coordinate system:

the 'positive' rotary direction for/in/of the coordinate plane
= df from the positive x-axis to the positive y-axis
around the origin proceeding first thru quadrant one
&
the 'negative' rotary direction for/in/of the coordinate plane
= df from the positive y-axis to the positive x-axis
around the origin proceeding first thru quadrant one

Δ the two definitions of
'positive' & 'negative' rotary directions for
a physical plane
&
a coordinate plane
agree
when
the coordinate axes are taken to be
in the canonical/standard pictorial position
viz



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∆ explaining the choice

• ¿ why should counterclockwise be chosen as positive ?

• 'counterclockwise' is equivalent to 'turning to the left'

• ¿ is there a built-in human bias
for turning left rather than right ?

• it is possible that for the majority of people
turning to the left
is physiologically easier than
turning to the right;
i find this true when riding on a bicycle;
i am right-handed

• oval racing courses for

bicycles
boats/ships
cars
dogs
humans on foot
horses
motorcycles
planes

all seem to be run counterclockwise

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☐ the measure in degrees / radians
= the degree / radian measure
of rotary angles

• a rotary angle is defined to be
positive / negative
and have a positive / negative numerical measure
according as its defining rotation is in the
counterclockwise / clockwise direction

• the absolute value of the measure of a rotary angle
in degrees / radians
may be determined by
(the possibly repeated use of)
the measurement of sectorial angles
in degrees / radians

• there is also a unit of measure of rotary angles called
turn $= 360^0 = 2\pi^r =$ once around the (counter) clock

☐ some nonnegative rotary angles
measured in degrees / radians / turns

- $0^o = 0^r =$ zero turns

- $90^o = \frac{\pi^r}{2} =$ quarter - turn

- $180^o = \pi^r =$ half - turn

- $270^o = \frac{3\pi^r}{2} =$ three - quarter - turn

- $360^o = 2\pi^r =$ one turn

- $720^o = 4\pi^r =$ two turns

- $1080^o = 6\pi^r =$ three turns

etc

□ the measure in radians = the radian measure
of radial angles

• by definition
the measure in radians = the radian measure
of a radial angle
is
the numerical value
of the length s of the intersection M of
the radial angle
and
the unit circle U
with center at the vertex of the radial angle
if the length of M exists;
if not
and
if M is Lebesgue measurable in dimension 1 on U,
then the numerical value
of the 1 - dimensional Lebesgue measure of M
may be used as the measure of the radial angle;
note that the radial angle is subtended by M

☐ the measure in steradians = the steradian measure
of solid angles

• by definition
the measure in steradians = the steradian measure
of a solid angle
is
the numerical value
of the area S of the intersection M of
the solid angle
and
the unit sphere U
with center at the vertex of the solid angle
if the area of M exists;
if not
and
if M is Lebesgue measurable in dimension 2 on U,
then the numerical value
of the 2 - dimensional Lebesgue measure of M
may be used as the measure of the solid angle;
note that the solid angle is subtended by M

□ notation for steradian measure

• the notation for steradian measure is

n steradians = n sr = n^{sr} wh n $\in$ nonneg real nr

□ some examples of the steradian measure of solid angles

• one octant solid angle $= \frac{\pi^{sr}}{2}$

• one quarter – turn dihedral solid angle $= \pi^{sr}$

• one flat solid angle $= 2\pi^{sr}$

• one three - quarter - turn dihedral solid angle $= 3\pi^{sr}$

• all space solid angle $= 4\pi^{sr}$

□ two analogous theorems,
one in the plane
and
one in space

• let a circle of radius r be centered
at the vertex of a radial plane angle
with radian measure α
and let circle and angle intersect in a set of length s;
then

$$s = r\alpha \ \& \ \alpha = \frac{s}{r}$$

• let a sphere of radius r be centered
at the vertex of a solid angle
with steradian measure α
and let sphere and angle intersect in a set of area S;
then

$$S = r^2\alpha \ \& \ \alpha = \frac{S}{r^2}$$

☐ two analogous corollaries
to the two preceding analogous theorems,
one in the plane
and
one in space

• an arc of length 1
on a circle of radius 1
subtends
a central plane angle of 1 radian

• a region of area 1
on a sphere of radius 1
subtends
a central solid angle of 1 steradian

□ measures of time

• by definition

measures of time include

1 day

= one mean solar day

= the (apparent) revolution period of the averaged Sun

1 day = 1 da = 1 d = 1^{da} = 1^{d}

= 24 hours = 24 hr = 24 h = 24^{hr} = 24^{h}

1 hour = 1 hr = 1 h = 1^{hr} = 1^{h}

= 60 minutes = 60 min = 60 m = 60^{min} = 60^{m}

1 minute = 1 min = 1 m = 1^{min} = 1^{m}

= 60 seconds = 60 sec = 60 s = 60^{sec} = 60^{s}

1 second = 1 sec = 1 s = 1^{sec} = 1^{s} = ca one heart beat

• the basic relationship between
the sexagesimal measure of time
and
the sexagesimal measure of angle;
by definition

$1 \text{ day} = 360^{o}$

because
the averaged Sun makes
exactly one (apparent) revolution
in one mean solar day
whence

1 day of time	=	1^{d}	=	360^{o}	=	360 degrees of angle
1 hour of time	=	1^{h}	=	15^{o}	=	15 degrees of angle
1 minute of time	=	1^{m}	=	$15'$	=	15 minutes of angle
1 second of time	=	1^{s}	=	$15''$	=	15 seconds of angle

☐ comparing notation for time and for angle

- time

$= A \text{ hours} + B \text{ minutes} + C \text{ seconds}$

$= A \text{ hr } B \text{ min } C \text{ sec}$

$= A \text{ h } B \text{ m } C \text{ s}$

$= A^{hr}\, B^{min}\, C^{sec}$

$= A^{h}\, B^{m}\, C^{s}$

- angle

$= A \text{ degrees} + B \text{ minutes} + C \text{ seconds}$

$= A \text{ deg } B \text{ min } C \text{ sec}$

$= A^{o}\, B^{/}\, C^{//}$

□ the standard equivalence
between angle measure & time measure

• angle ito time

$360^{o} = 24^{h}$

$270^{o} = 18^{h}$

$180^{o} = 12^{h}$

$90^{o} = 6^{h}$

$1^{o} = 4^{m}$

$1' = 4^{s}$

$1'' = \left(\frac{1}{15}\right)^{s}$

- time ito angle

$24^h = 360^o$

$18^h = 270^o$

$12^h = 180^o$

$6^h = 90^o$

$1^h = 15^o$

$1^m = 15^/$

$1^s = 15^{//}$

□ the seven special (= integer - hour) angles in quadrant I ito
degrees, radians, hours

since $1^h = 15^o = \frac{\pi^r}{12}$
it follows that

- $0^o = 0^r = 0^h$
- $15^o = \frac{\pi^r}{12} = 1^h$
- $30^o = \frac{\pi^r}{6} = 2^h$
- $45^o = \frac{\pi^r}{4} = 3^h$
- $60^o = \frac{\pi^r}{3} = 4^h$
- $75^o = \frac{5\pi^r}{12} = 5^h$
- $90^o = \frac{\pi^r}{2} = 6^h$

☐ an illustration of esthetics
as a guiding principle in mathematics

• a pretty formula
defines radian measure of a sectorial angle;
think of a given circle
with a given central angle
intercepting a subtending arc on the circle;
think of the associated numbers
viz
the length of the radius $= r$
the length of the subtending circle arc $= s$
the measure of the central angle $= \alpha$ (to be determined)
and note the alphabetic choices of initial letters of words;
then the following formula holds
$s = r\alpha$
if and only if
$\alpha = \frac{s}{r}$
which is the definition
of the radian measure of the central angle;

no other definition of angle measure will do
to make the desired formula hold;
eg
if the angle α were measured in degrees,
then the formula would become

$$s = \frac{\pi}{180} r \alpha$$

which is not so pretty
nor so simple nor so efficient nor so theoretically useful;
one can say that radian measure of angles
is natural and inevitable
while degree measure of angles
is an historical accident
altho it is now well - entrenched in our lives
and we have to make the best of it
(but it is not too bad at that)

□ historical comments

• we have the ancient Babylonians to thank
for our present use of sexagesimal (= base 60) numeration
in the measure of angle/arc/time
which we inherited from them;
ca 2000 BCE the Babylonians in Mesopotamia,
between the Tigris & Euphrates rivers in the Middle East,
began to use a number system based on 60;
they likely chose base 60 because of
the relatively large number of factors of 60
which has the effect of producing lots
of easily written fractions

• the ancient Babylonians & Greeks
divided up the lighter day time into 12 equal parts
& also the darker night time into 12 equal parts;
12 was likely chosen because
12 has many factors
& 12 is still relatively small
& 12 is also itself a factor of 60

• it is an historical accident that
the degree of angle was chosen so that
there are 360 degrees in a round angle;
it was likely caused by taking
the angle of an equilateral triangle
viz 60 degrees
as the unit of angle
and then using the base 60 numeration system
so that a degree is recognized as
one-sixtieth of an angle
of an equilateral triangle

• ca 150 CE
Claudius Ptolemy (Latin: Ptolemaeus) of Alexandria
(ca 90 - 168 CE, Egyptian;
astronomer, geographer, mathematician)
used characters on numbers with sexagesimal fractions
that resemble our present-day characters for
degree, minute, second of angle/arc

• in the 16th century
the characters
^{o}, $^{\prime}$, $^{\prime\prime}$, $^{\prime\prime\prime}$, ^{iv}, etc
(the first three are our familiar ones)
began to be used
as righthand superscripts
denoting
degrees, minutes, seconds, thirds, fourths, etc
in sexagesimal numeration

• the above symbols may be simply designations of
0, 1, 2, 3, 4, etc
which with negative signs are the exponents of 60
in a number given in units of degrees
with the fractional part written to the base 60

□ etymology of words
for the measures of angle/arc/time

• degree
from
degradus (Latin) = degree
from
de (Latin) = down
+
gradus (Latin) = grade, step

• hour
from
hora (Latin) = hour
from
'ωρα (Greek) = hour

• minute
from
minuta (Latin) = small
from
pars minuta prima (Latin) = first small part

• second
from
secunda (Latin) = second
from
pars minuta secunda (Latin) = second small part

• radian

from

radius (English)

from

radius (Latin) = spoke of a wheel

• steradian

from

ste- = abbreviation of

stereo- (English) = solid

+

radian (English)

• sexagesimal = relating to the number sixty

from

sexigesimus (Latin) = sixtieth

from

sexaginta (Latin) = sixty

• clockwise = rotating like
the hands of a clock
from
clock (English)
+
-wise (English) = in the manner of

• counterclockwise = rotating contrary to
the hands of a clock
from
counter- (English) = opposite
+
clockwise (English)

• the word 'radian' first appeared in print in 1879;
'radian' comes from
the Latin-English word
radius minus -us plus -an;
the Latin word 'radius'
means many things
viz
beam of light
bone in human forearm
kind of long olive
measuring rod
radius of a circle
ray
rod
spoke of a wheel
staff
stake
weaving shuttle

• the word 'steradian' first appeared in print in 1881;
the prefix 'ste-' comes from
the English prefix 'stereo-' = solid
which comes from the Greek word
στερεος = solid

• a wild guess:
radius + angle
= radi + an
= radian