The Inverse Function Tableau

#2 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2000

500 Angell St #414
Providence RI 02906

GG2-2

☐ the inverse function tableau

• the inverse function tableau = IFT
is a simple arrangement of text that displays
the conceptual determination
of the inverse function of a given function
or the end result of a computational determination
of the inverse function of a given function;
if the inverse function is multiple - valued,
then a principal - valued inverse function
may be included;
IFT is, in particular, a convenient device to indicate
the notation, domains, and ranges
of the functions involved

• to construct the IFT of a function

$y = f(x)$, read ' y equals f of x' ,

solve for x in terms of y

(where ' solving' may mean

more of a thought than an algorithm)

to obtain an equivalent equation $x = f^{-1}(y)$,

and then interchange x and y

to make

x the independent variable

and

y the dependent variable,

thus obtaining the inverse function

$y = f^{-1}(x)$, read ' y equals f inverse of x' ,

with the original notation

for independent and dependent variables;

this procedure may be indicated schematically
as follows:

$y = f(x) \quad \Leftrightarrow \quad x = f^{-1}(y)$

$x = f(y) \quad \Leftrightarrow \quad y = f^{-1}(x)$

where the mutually inverse functions,
with the same notation
for independent and dependent variables,
appear in
the upper left corner and the lower right corner;
if the inverse function is multiple - valued,
then functional values may be chosen to form
a principle - valued inverse function
which is a single - valued function
with the same domain;
note that the inverse function of the inverse function
is the original function
ie the inverse function of $y = f^{-1}(x)$ is $y = f(x)$;
results are arranged in a certain way
for oversight and insight;
various examples are given below

☐ notation and geometric names
for the five basic number systems
of classical analysis

(0) $\mathbb{R}$, $\overline{\mathbb{R}}$, $\dot{\mathbb{R}}$, $\mathbb{C}$, $\dot{\mathbb{C}}$

(1) the real line
$=_{df}$ the set of all real numbers
$=_{dn}$ $\mathbb{R}$
$=_{rd}$ (open cap) ar

(2) the extended real line
$=_{df}$ $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$
$=_{dn}$ $\overline{\mathbb{R}}$
$=_{rd}$ (open cap) ar bar

(3) the projective real line
$=_{df}$ $\mathbb{R} \cup \{\infty\}$
$=_{dn}$ $\dot{\mathbb{R}}$
$=_{rd}$ (open cap) ar dot

(4) the complex plane

$=_{df}$ the set of all complex numbers

$=_{dn}$ $\mathbb{C}$

$=_{rd}$ (open cap) cee

(5) the complex sphere

$=_{df}$ $\mathbb{C} \cup \{\infty\}$

$=_{dn}$ $\dot{\mathbb{C}}$

$=_{rd}$ (open cap) cee dot

☐ some convenient abbreviations

- function = fcn
- domain = dmn
- range = rng
- inverse = inv
- self - inverse = si
- single - valued = sv
- double - valued = dv
- multiple - valued = mv
- infinitely - many - valued = ∞v
- principal - valued = pv

• variable = var

• real number variable
= real variable = real var

• complex number variable
= complex variable = complex var

• variable which ranges over items
= item variable = item var

• the set of all variables whose range is the set S
= var S

• the downward arrow $\downarrow$ may be read
' has a / an / the'

☐ E1. the general real linear function

IFT $y = ax + b \ \ (x \in \mathbb{R})$

$(a, b \in \mathbb{R} \ \& \ a \neq 0) \ \ (x, y \in \text{real var})$

• fcn

$y = ax + b$

dmn: $-\infty < x < +\infty$

rng: $-\infty < y < +\infty$

$\downarrow$

• sv inv fcn

$$y = -\frac{1}{a}x - \frac{b}{a}$$

dmn: $-\infty < x < +\infty$

rng: $-\infty < y < +\infty$

□ E2. the general complex linear function

IFT $w = az + b$ $(z \in \mathbb{C})$

$(a, b \in \mathbb{C}$ & $a \neq 0)$ $(z, w \in$ complex var$)$

• fcn

$w = az + b$

dmn: $z \in \mathbb{C}$

rng: $w \in \mathbb{C}$

↓

• sv inv fcn

$$w = -\frac{1}{a}z - \frac{b}{a}$$

dmn: $z \in \mathbb{C}$

rng: $w \in \mathbb{C}$

☐ E3. the real square function

IFT $y = x^2 \quad (x \in \mathbb{R})$
$(x, y \in \text{real var})$

• fcn

$y = x^2$
dmn: $-\infty < x < +\infty$
rng: $0 \le y < +\infty$
↓
• dv inv fcn

$y = \pm\sqrt{x}$
dmn: $0 \le x < +\infty$
rng: $-\infty < y < +\infty$
↓
• pv inv fcn

$y = \sqrt{x}$
dmn: $0 \le x < +\infty$
rng: $0 \le y < +\infty$

☐ E4. the complex square function

IFT $w = z^2 \quad (z \in \mathbb{C})$

$(z, w \in \text{complex var})$

• fcn

$w = z^2$

dmn: $z \in \mathbb{C}$

rng: $w \in \mathbb{C}$

$\downarrow$

• dv inv fcn

$w = \pm\sqrt{z}$

dmn: $z \in \mathbb{C}$

rng: $w \in \mathbb{C}$

$\downarrow$

• pv inv fcn

$w = \sqrt{z}$

dmn: $z \in \mathbb{C}$

rng: $0 \le \operatorname{Ang} w < \pi$

☐ E5. the real exponential function

IFT $y = e^{x}$ $(x \in \mathbb{R})$

$(x, y \in \text{real var})$

• fcn

$y = e^{x}$

dmn: $-\infty < x < +\infty$

rng: $0 < y < +\infty$

$\downarrow$

• sv inv fcn

$y = \log x = \ln x$

dmn: $0 < x < +\infty$

rng: $-\infty < y < +\infty$

☐ E6. the complex exponential function

IFT $w = e^z$ $(z \in \mathbb{C})$
$(z, w \in \text{complex var})$

• fcn
$w = e^z$
dmn: $z \in \mathbb{C}$
rng: $w \in \mathbb{C}\ \&\ w \neq 0$
$\downarrow$
• ∞v inv fcn
$w = \log z$
dmn: $z \in \mathbb{C}\ \&\ z \neq 0$
rng: $w \in \mathbb{C}$
$\downarrow$
pv inv fcn
$w = \operatorname{Log} z$
dmn: $z \in \mathbb{C}\ \&\ z \neq 0$
rng: $w \in \mathbb{C}\ \&\ 0 \leq \operatorname{Im} w < 2\pi$

for $0 \neq z \in \mathbb{C}$:
$\log z = \ln|z| + i \operatorname{ang} z = \ln|z| + i(\operatorname{Ang} z + 2\pi\mathbb{Z})$
$\operatorname{Log} z = \ln|z| + i \operatorname{Ang} z$ wh $0 \leq \operatorname{Ang} z < 2\pi$

☐ E7. the real sine function

IFT $y = \sin x \quad (x \in \mathbb{R})$
$(x, y \in \text{real var})$

• fcn
$y = \sin x$
dmn: $-\infty < x < +\infty$
rng: $-1 \le y \le 1$
↓
• ∞v inv fcn
$y = \sin^{-1} x$
dmn: $-1 \le x \le 1$
rng: $-\infty < y < +\infty$
↓
• pv inv fcn
$y = \mathrm{Sin}^{-1} x$
dmn: $-1 \le x \le 1$
rng: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

☐ E8. the complex sine function

IFT $w = \sin z \ (z \in \mathbb{C})$
$(z, w \in \text{complex var})$

• fcn
$w = \sin z$
dmn: $z \in \mathbb{C}$
rng: $w \in \mathbb{C}$
$\downarrow$
• ∞v inv fcn
$w = \sin^{-1} z$
dmn: $z \in \mathbb{C}$
rng: $w \in \mathbb{C}$
$\downarrow$
• pv inv fcn
$$w = \mathrm{Sin}^{-1} z = \frac{1}{i} \ \mathrm{Log}\,(iz + \sqrt{1 - z^2}\ )$$
dmn: $z \in \mathbb{C}$
rng: set of all w as above

☐ E9. some self - inverse functions

• the real negation function
$f(x) = -x$
on the real line
or
on the extended real line
or
on the projective real line

• the complex negation function
$f(z) = -z$
on the complex plane
or
on the complex sphere

• the real reciprocal function

$f(x) = \frac{1}{x}$

on the punctured real line

or

on the projective real line

• the complex reciprocal function

$f(z) = \frac{1}{z}$

on the punctured complex plane

or

on the complex sphere

• the complex conjugation function

$f(z) = \overline{z}$

on the complex plane

or

on the complex sphere

• any combination of

negation, reciprocation, conjugation

- transposition of matrices
- conjugation of matrices
- conjugate transposition of matrices
- inversion of matrices
- formation of dual spaces of vector spaces
- inversion in any group
- negation in any additive group
- reciprocation in any field

- complementation of sets

- conversion of relations

- passage to the dual in any duality theory

- passage to the converse in logic

- passage to the contrapositive in logic

- interchange of two specified items in an array

- reflection in a point / line / plane / etc

- inversion of functions
ie the inverse function of the inverse function
is the original function

- etc

□ E10. the real Joukowski transformation

IFT $y = \frac{1}{2}(x + \frac{1}{x})$ (x ∈ ☼̇)

$(x, y \in \text{var}\, \dot{\mathbb{R}})$

• fcn

$y = \frac{1}{2}(x + \frac{1}{x})$

dmn: $x \in \dot{\mathbb{R}}$

rng: $-\infty < y \le -1 \;\vee\; 1 \le y < +\infty \;\vee\; y = \infty$

↓

dv inv fcn

$y = x \pm \sqrt{x^2 - 1}$

dmn: $-\infty < x \le -1 \;\vee\; 1 \le x < +\infty \;\vee\; x = \infty$

rng: $y \in \dot{\mathbb{R}}$

↓

pv inv fcn

$y = x + \sqrt{x^2 - 1}$

dmn: $-\infty < x \le -1 \;\vee\; 1 \le x < +\infty \;\vee\; x = \infty$

rng: $-1 \le y \le 0 \;\vee\; 1 \le y < +\infty \;\vee\; y = \infty$

□ E11. the complex Joukowski transformation

IFT $w = \frac{1}{2}(z + \frac{1}{z})$ $(z \in \dot{\mathbb{C}})$

$(z, w \in \text{var}\, \dot{\mathbb{C}})$

fcn

$w = \frac{1}{2}(z + \frac{1}{z})$

dmn: $z \in \dot{\mathbb{C}}$

rng: $w \in \dot{\mathbb{C}}$

↓

dv inv fcn

$w = z \pm \sqrt{z^2 - 1}$

dmn: $z \in \dot{\mathbb{C}}$

rng: $w \in \dot{\mathbb{C}}$

☐ E12. the general real homography

IFT $y = \dfrac{ax+b}{cx+d}$ $(x \in \dot{\mathbb{R}})$

$(a, b, c, d \in \mathbb{R}$ & $ad - bc \neq 0)$ $(x, y \in \text{var}\ \dot{\mathbb{R}})$

• fcn

$$y = \frac{ax+b}{cx+d}$$

dmn: $x \in \dot{\mathbb{R}}$

rng: $y \in \dot{\mathbb{R}}$

↓

• sv inv fcn

$$y = \frac{dx-b}{-cx+a}$$

dmn: $x \in \dot{\mathbb{R}}$

rng: $y \in \dot{\mathbb{R}}$

☐ E13. the general complex homography

IFT $w = \dfrac{az+b}{cz+d} \quad (z \in \dot{\mathbb{C}})$

$(a, b, c, d \in \mathbb{C}$ & $ad - bc \neq 0)$ $(z, w \in \text{var}\,\dot{\mathbb{C}})$

• fcn

$$w = \frac{az+b}{cz+d}$$

dmn: $z \in \dot{\mathbb{C}}$

rng: $w \in \dot{\mathbb{C}}$

↓

• sv inv fcn

$$w = \frac{dz-b}{-cz+a}$$

dmn: $z \in \dot{\mathbb{C}}$

rng: $w \in \dot{\mathbb{C}}$

□ E14. the gudermannian

IFT $y = \mathrm{gd}\, x \quad (x \in \mathbb{R})$
$(x, y \in \text{real var})$

• fcn

$$y = \mathrm{gd}\, x = \mathrm{Tan}^{-1} \sinh x = \int_0^x \mathrm{sech}\, t\, dt$$

dmn: $-\infty < x < +\infty$

rng: $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

$\downarrow$

• sv inv fcn

$$y = \mathrm{gd}^{-1} x = \sinh^{-1} \tan x = \int_0^x \sec t\, dt = \ln(\sec x + \tan x)$$

dmn: $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

rng: $-\infty < y < +\infty$

• ¿was ist gut about the gudermannian?

one good thing (there are others) is that

it provides a real bridge between

the trigonometric functions

&

the hyperbolic functions

viz

$u = \mathrm{gd}\, x$

$\Rightarrow$

$\sin u = \tanh x$

$\cos u = \mathrm{sech}\, x$

$\tan u = \sinh x$

$\cot u = \mathrm{csch}\, x$

$\sec u = \cosh x$

$\csc u = \coth x$

☐ E15. two especially notable examples of inversion

• The Fundamental Theorem of Calculus
states that
differentiation & integration
are inverse operations;
what one does, the other undoes

• elliptic functions & elliptic integrals
are inverse functions of each other
(with lots of technical details)

□ historical note

• inspired by the example of the inverse
elliptic functions & elliptic integrals
& attendant insights following from
the recognition of this inverse relationship,
Jacobi said
Du muss immer umkehren! (German)
= Thou must always invert!
thus proclaiming that unremitting inversion
is the secret of success in mathematics