The Algebra of Real 3-Vectors

#98 of Gottschalk's Gestalts

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GG98-2

C. we are here considering
three - dimensional real coordinate vectors
as an introductory account
of vector / linear algebra
in regard to the basic linear operations
in a vector space,
operating on the premise that
it is good for the understanding
to build up a generous supply
of examples and special cases;
we do not consider linear transformations
between vector spaces in this context,
thinking that the best approach to that topic
is to emphasize the direction of
from the general to the particular

D. scalars

• a scalar
$=_{df}$ a real number
= an element of $\mathbb{R}$

• scalar variables are taken to be
certain lowercase Greek letters
as α,β,γ
which will be clear from context

D. vectors

• a (three - dimensional real coordinate) vector
$=_{df}$ an ordered triple of real numbers
$=$ an element of $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$

• vector variables are taken to be
certain bold - faced lowercase English letters
as $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$
which will be clear from context

• the first / second / third component or coordinate
of a vector **a**
$=_{df}$ the first / second / third term
of the ordered triple **a**
$=_{dn}$ the corresponding light - faced letter a
with subscript 1 / 2 / 3
eg
$\mathbf{a} = (a_1, a_2, a_3)$
$\mathbf{b} = (b_1, b_2, b_3)$
etc

• the (three - dimensional real coordinate) vector space

$=_{dn}$ V

$=_{df}$ the set of all vectors

= the set of all ordered triples of real numbers

$= \{(x,y,z) | x,y,z \in \mathbb{R}\}$

$= \mathbb{R}^3$

note:

V is the capital letter vee used in earlier times

as a letter for the vowel Uu as well as the consonant Vv;

the vowel is the initial letter of 'universe';

it is at least reinforcement that V is also

the capitalized initial letter of the word 'vector'

D. arrows

• an arrow

$=_{df}$ a directed straight line segment

whose initial point is called its 'tail'

&

whose terminal point is called its 'tip'

• the geometric point with coordinate - triple **a** in $\mathbb{R}^3$

$=_{ab}$ the point **a**

• the (canonical) arrow for $\mathbf{a} \neq \mathbf{0}$

$=_{dn}$ $\vec{\mathbf{a}}$

$=_{rd}$ **a** arrow = **a** air ('arrow' pronounced rapidly)

$=_{df}$ the arrow

from the origin $\mathbf{0} = (0,0,0)$

to the point $\mathbf{a} = (a_1, a_2, a_3)$

C. an arrow is uniquely characterized by
its length = magnitude
&
its direction

C. the algebraic vector **a**
(which is an ordered triple of real numbers)
&
the geometric vector $\vec{\mathbf{a}}$
(which is a directed line segment)
may be said to be different aspects
of the same thing;
this suggests the notion & investigation of
axiomatically defined abstract general vector spaces
would be clarifying and unifying;
and indeed that is the case

C. sometimes the notational distinction between
a and $\vec{\mathbf{a}}$ is ignored

C. there is a uniquely defined natural
one - to - one correspondence between
the algebraic nonzero vectors as ordered number triples
&
the geometric arrows all issuing from the origin;
¿what happened to the zero vector **0**
which is the algebraic name of the origin ?
properly speaking there is no arrow asssigned to **0**;
however sometimes it is convenient
to posit a null arrow $\vec{\mathbf{0}}$ assigned to the zero vector **0**;
it is interesting that the zero vector
is so important and centrally located
in the algebraic POV
and yet in the geometric POV
the null arrow seems negligible
and comes up for mandatory consideration only after
the algebra is introduced;
it is clear that the null arrow should have length zero
but that it has no uniquely definable direction;
the null arrow should be considered to have
either no direction or all directions

D. the zero vector

• the zero vector

$=_{dn}$ **0**

$=_{rd}$ (vector) oh / zero

$=_{df}$ (0,0,0)

• vector zero is componentwise scalar zero

BP. zero vector

- $\mathbf{0} \in$ invariant under negation

$-\mathbf{0} = \mathbf{0}$

- $\mathbf{0} =$ the unique vector invariant under negation

$-\mathbf{a} = \mathbf{a} \Leftrightarrow \mathbf{a} = \mathbf{0}$

- $\mathbf{0} =$ the bilateral additive identity element

$\mathbf{a} + \mathbf{0} = \mathbf{a} = \mathbf{0} + \mathbf{a}$

- $\mathbf{0} =$ the right subtractive identity element

$\mathbf{a} - \mathbf{0} = \mathbf{a}$

- $\mathbf{0}$ = the right scalector - multiplicative nullity element

$\alpha\mathbf{0} = \mathbf{0}$

- 0 = the left scalector - multiplicative nullity element

$0\mathbf{a} = \mathbf{0}$

- $\alpha\mathbf{a} = \mathbf{0} \Leftrightarrow \alpha = 0 \vee \mathbf{a} = \mathbf{0}$

- $\mathbf{0}$ = the bilateral dot - product nullity element

$\mathbf{a} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{a}$

- $\mathbf{0}$ = the bilateral cross - product nullity element

$\mathbf{a} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{a}$

- $\mathbf{0}$ nullifies any determinant

$|\mathbf{a}, \mathbf{b}, \mathbf{0}| = |\mathbf{a}, \mathbf{0}, \mathbf{b}| = |\mathbf{0}, \mathbf{a}, \mathbf{b}| = 0$

- $\mathbf{0} - \mathbf{a} = -\mathbf{a}$

- $\mathbf{a} - \mathbf{a} = \mathbf{0}$

- $(\forall \mathbf{x} . \mathbf{a} \cdot \mathbf{x} = 0) \Leftrightarrow \mathbf{a} = \mathbf{0}$

- $(\forall \mathbf{x} . \mathbf{a} \times \mathbf{x} = \mathbf{0}) \Leftrightarrow \mathbf{a} = \mathbf{0}$

D. the negation of a vector

• the negation / negate of $\mathbf{a} = (a_1, a_2, a_3)$

$=_{dn} \ -\mathbf{a}$

$=_{rd}$ minus $\mathbf{a}$

$=_{df} \ (-a_1, -a_2, -a_3)$

$\in$ vector

• vector negation is a unary operation in V

$-: V \to V$

$\quad \mathbf{a} \mapsto -\mathbf{a}$

• vector negation is componentwise scalar negation;
to negate a vector
negate each component

BP. vector negation

- $-\mathbf{0} = \mathbf{0}$
- $-(-\mathbf{a}) = \mathbf{a}$ (law of double negation)
- $-(\mathbf{a} + \mathbf{b}) = -\mathbf{a} - \mathbf{b}$
- $-(\mathbf{a} - \mathbf{b}) = -\mathbf{a} + \mathbf{b}$
- $-(\alpha\mathbf{a}) = (-\alpha)\mathbf{a} = \alpha(-\mathbf{a}) = -\alpha\mathbf{a}$
- $-(\mathbf{a} \cdot \mathbf{b}) = (-\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (-\mathbf{b})$
- $|-\mathbf{a}| = |\mathbf{a}|$
- $-(\mathbf{a} \times \mathbf{b}) = (-\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (-\mathbf{b})$
- $-|\mathbf{a}, \mathbf{b}, \mathbf{c}| = |-\mathbf{a}, \mathbf{b}, \mathbf{c}| = |\mathbf{a}, -\mathbf{b}, \mathbf{c}| = |\mathbf{a}, \mathbf{b}, -\mathbf{c}|$

- vector negation ito scalector multiplication

$-\mathbf{a} = (-1)\mathbf{a}$

D. the sum of two vectors

• the sum of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$

$=_{dn}$ $\mathbf{a} + \mathbf{b}$

$=_{rd}$ **a** plus **b**

$=_{df}$ $(a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$

$\in$ vector

• vector addition is a binary operation in V

$+ : V \times V \to V$

$\quad (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} + \mathbf{b}$

• vector addition is componentwise scalar addition;
to add two vectors
add corresponding components

BP. vector addition

• addition $\in$ commutative

$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

• addition $\in$ associative

$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

• $\mathbf{0}$ = the bilateral additive identity element

$\mathbf{a} + \mathbf{0} = \mathbf{a} = \mathbf{0} + \mathbf{a}$

• $-\mathbf{a}$ = the bilateral additive inverse element of $\mathbf{a}$

$\mathbf{a} + (-\mathbf{a}) = \mathbf{0} = (-\mathbf{a}) + \mathbf{a}$

• $(\mathrm{V}, \mathbf{0}, -, +) \in$ additive abelian group

- additive combinations of **a** and **0**

$\mathbf{a} + \mathbf{a} = 2\mathbf{a}$

$\mathbf{a} + \mathbf{0} = \mathbf{a}$

$\mathbf{0} + \mathbf{a} = \mathbf{a}$

$\mathbf{0} + \mathbf{0} = \mathbf{0}$

- combinations of negation, addition, subtraction

$(-\mathbf{a}) + \mathbf{b} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a} = -(\mathbf{a} - \mathbf{b})$

$\mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b}$

$(-\mathbf{a}) + (-\mathbf{b}) = -\mathbf{a} - \mathbf{b} = -(\mathbf{a} + \mathbf{b})$

- iterated addition ito scalector multiplication

$\mathbf{0} = 0\mathbf{a}$

$\mathbf{a} = 1\mathbf{a}$

$\mathbf{a} + \mathbf{a} = 2\mathbf{a}$

$\mathbf{a} + \mathbf{a} + \mathbf{a} = 3\mathbf{a}$

etc

$\mathbf{a} + \mathbf{a} + \cdots + \mathbf{a}$ (n terms) $= \mathbf{na}$

wh n $\in$ nonneg int

D. the difference of two vectors

• the difference of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$

$=_{dn}$ $\mathbf{a} - \mathbf{b}$

$=_{rd}$ **a** minus **b**

$=_{df}$ $(a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3)$

$\in$ vector

• vector subtraction is a binary operation in V

$-: V \times V \to V$

$(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} - \mathbf{b}$

• vector subtraction is componentwise scalar subtraction;
to subtract two vectors
subtract corresponding components

BP. vector subtraction

• subtraction $\in$ expressible ito addition & negation

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

• addition $\in$ expressible ito subtraction & negation

$$\mathbf{a} + \mathbf{b} = \mathbf{a} - (-\mathbf{b})$$

• negation $\in$ expressible ito subtraction & the vector zero

$$-\mathbf{a} = \mathbf{0} - \mathbf{a}$$

• the difference between a vector and itself = the vector zero

$$\mathbf{a} - \mathbf{a} = \mathbf{0}$$

• subtraction $\in$ anticommutative

$$\mathbf{a} - \mathbf{b} = -(\mathbf{b} - \mathbf{a})$$

• subtractive combinations of **a** and **0**

$\mathbf{a} - \mathbf{a} = \mathbf{0}$

$\mathbf{a} - \mathbf{0} = \mathbf{a}$

$\mathbf{0} - \mathbf{a} = -\mathbf{a}$

$\mathbf{0} - \mathbf{0} = \mathbf{0}$

• combinations of negation, addition, subtraction

$(-\mathbf{a}) - \mathbf{b} = -\mathbf{a} - \mathbf{b} = -(\mathbf{a} + \mathbf{b})$

$\mathbf{a} - (-\mathbf{b}) = \mathbf{a} + \mathbf{b}$

$(-\mathbf{a}) - (-\mathbf{b}) = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a} = -(\mathbf{a} - \mathbf{b})$

• iterated subtraction ito scalector multiplication

$-\mathbf{a} = (-1)\mathbf{a} = -1\mathbf{a}$

$-\mathbf{a} - \mathbf{a} = (-2)\mathbf{a} = -2\mathbf{a}$

$-\mathbf{a} - \mathbf{a} - \mathbf{a} = (-3)\mathbf{a} = -3\mathbf{a}$

etc

$-\mathbf{a} - \mathbf{a} - \cdots - \mathbf{a}$ (n terms) $= (-n)\mathbf{a} = -n\mathbf{a}$

wh $n \in$ pos int

(here negation is 'unary subtraction')

• the binary operations of
vector addition
&
vector subtraction
are inverses of each other

$$(\mathbf{a} + \mathbf{b}) - \mathbf{b} = \mathbf{a}$$

$$(\mathbf{a} - \mathbf{b}) + \mathbf{b} = \mathbf{a}$$

• iterated subtraction
is reducible to a single subtraction

$$(\mathbf{a} - \mathbf{b}) - \mathbf{c} = \mathbf{a} - (\mathbf{b} + \mathbf{c})$$

$$((\mathbf{a} - \mathbf{b}) - \mathbf{c}) - \mathbf{d} = \mathbf{a} - (\mathbf{b} + \mathbf{c} + \mathbf{d})$$

etc

D. the left scalector product
of a scalar and a vector

• the left scalector product of α and $\mathbf{a} = (a_1, a_2, a_3)$
= the scalector product of α and **a**
= the product of α and **a**
$=_{dn}$ $\alpha\mathbf{a}$
$=_{rd}$ α times **a** = α **a**
$=_{df}$ $(\alpha a_1, \alpha a_2, \alpha a_3)$
$\in$ vector

• left scalector multiplication is a function
$\mathbb{R} \times V \to V$
$(\alpha, \mathbf{a}) \mapsto \alpha\mathbf{a}$

• left scalector multiplication is
componentwise scalar multiplication;
to multiply a scalar and a vector
multiply the scalar and the components of the vector

D. the right scalector product
of a scalar and a vector

• the right scalector product of α and $\mathbf{a} = (a_1, a_2, a_3)$

$=$ the scalector product of **a** and α

$=$ the product of **a** and α

$=_{dn}$ $\mathbf{a}\alpha$

$=_{rd}$ **a** times α $=$ $\mathbf{a}\ \alpha$

$=_{df}$ $(a_1\alpha,\ a_2\alpha,\ a_3\alpha)$

$\in$ vector

• right scalector multiplication is a function

$V \times \mathbb{R} \to V$

$(\mathbf{a},\alpha) \mapsto \mathbf{a}\alpha$

• right scalector multiplication is
componentwise scalar multiplication;
to multiply a vector and a scalar
multiply the components of the vector and the scalar

D. the bilateral scalector product
of two scalars and a vector

• the bilateral scalector product of α and β with $\mathbf{a} = (a_1, a_2, a_3)$
= the scalector product of α and **a** and β
= the product of α and **a** and β
$=_{dn}$ $\alpha \mathbf{a} \beta$
$=_{rd}$ α times **a** times β = α **a** β
$=_{df}$ $(\alpha a_1 \beta, \alpha a_2 \beta, \alpha a_3 \beta)$
$\in$ vector

• bilateral scalector multiplication is a function
$\mathbb{R} \times V \times \mathbb{R} \rightarrow V$
$(\alpha, \mathbf{a}, \beta) \mapsto \alpha \mathbf{a} \beta$

• bilateral scalector multiplication is
componentwise scalar multiplication;
to multiply a scalar and a vector and a scalar
multiply the first scalar
and the components of the vector
and the second scalar

N. in the present context
there is no essential distinction among
left and right and bilateral scalector multiplications;
there is only a notational distinction
since they are virtually interchangeable
because of the commutativity
of real number multiplication;
on occasion it may be convenient
to recognize the ability to multiply
the present vectors on either side or on both sides;
however for generalizations that involve
noncommutative multiplication as in division rings,
the distinction is essential;
there are then
  left vector spaces
& right vector spaces
& bilateral (= two - sided) vector spaces

N. the manufactured word 'scalector'
comes from a blend of
'scalar' and 'vector'
viz
scalector $\leftarrow$ <u>scal</u>ar + v<u>ector</u>;
the phrase 'scalar multiplication'
should be used only to refer to
the dot - product of two vectors
and not the product of a scalar and a vector;
it is easier & shorter to say & write
scalector multiplication / product
than
scalar - vector multiplication / product

BP. scalector multiplications

• all three scalector multiplications are distributive over vector addition & subtraction

$$\alpha(\mathbf{a} \pm \mathbf{b}) = \alpha\mathbf{a} \pm \alpha\mathbf{b}$$

$$(\mathbf{a} \pm \mathbf{b})\beta = \mathbf{a}\beta \pm \mathbf{b}\beta$$

$$\alpha(\mathbf{a} \pm \mathbf{b})\beta = \alpha\mathbf{a}\beta \pm \alpha\mathbf{b}\beta$$

• all three scalector multiplications are distributive over scalar addition & subtraction

$$(\alpha \pm \beta)\mathbf{a} = \alpha\mathbf{a} \pm \beta\mathbf{a}$$

$$\mathbf{a}(\gamma \pm \delta) = \mathbf{a}\gamma \pm \mathbf{a}\delta$$

$$(\alpha \pm \beta)\mathbf{a}\gamma = \alpha\mathbf{a}\gamma \pm \beta\mathbf{a}\gamma$$

$$\alpha\mathbf{a}(\gamma \pm \delta) = \alpha\mathbf{a}\gamma \pm \alpha\mathbf{a}\delta$$

• hereinafter we consider only
left scalector multiplication
and call it THE scalector multiplication

• scalector multiplication $\in$ associative
$(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a})$

• scalector multiplication nullity laws
$0\mathbf{a} = \mathbf{0}$
$\alpha\mathbf{0} = \mathbf{0}$
$\alpha\mathbf{a} = \mathbf{0} \Leftrightarrow \alpha = 0 \vee \mathbf{a} = \mathbf{0}$

• scalector multiplication unity law
$1\mathbf{a} = \mathbf{a}$

D. the dot product of two vectors

• the dot product of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$

$=_{dn}$ $\mathbf{a} \cdot \mathbf{b}$

$=_{rd}$ **a** dot **b**

$=_{df}$ $a_1 b_1 + a_2 b_2 + a_3 b_3$

$\in$ scalar

• scalar / inner multiplication is a function

$\cdot : V \times V \to \mathbb{R}$

$(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \cdot \mathbf{b}$

• to form the dot product

multiply corresponding components

& add

BP. dot product

• $\mathbf{0}$ factor nullifies dot product;
dot product vanishes if a factor vanishes

$\mathbf{a} \cdot \mathbf{0} = 0$

$\mathbf{0} \cdot \mathbf{a} = 0$

• dot product $\in$ commuative

$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

• negation floats in a dot product

$(-\mathbf{a}) \cdot \mathbf{b} = -\mathbf{a} \cdot \mathbf{b}$

$\mathbf{a} \cdot (-\mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$

• double negation disappears in a dot product

$(-\mathbf{a}) \cdot (-\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$

• dot product $\in$ bihomogeneous

$(\alpha\mathbf{a})\cdot\mathbf{b} \;=\; \alpha(\mathbf{a}\cdot\mathbf{b})$

$\mathbf{a}\cdot(\beta\mathbf{b}) \;=\; \beta(\mathbf{a}\cdot\mathbf{b})$

$(\alpha\mathbf{a})\cdot(\beta\mathbf{b}) \;=\; \alpha\beta(\mathbf{a}\cdot\mathbf{b})$

• dot product $\in$ bilaterally distributive
over addition & subtraction

$\mathbf{a}\cdot(\mathbf{b}\pm\mathbf{c}) \;=\; \mathbf{a}\cdot\mathbf{b}\pm\mathbf{a}\cdot\mathbf{c}$

$(\mathbf{a}\pm\mathbf{b})\cdot\mathbf{c} \;=\; \mathbf{a}\cdot\mathbf{c}\pm\mathbf{b}\cdot\mathbf{c}$

• dot product $\in$ bilinear ie linear in each factor

$(\alpha\mathbf{a}+\beta\mathbf{b})\cdot\mathbf{c} \;=\; \alpha(\mathbf{a}\cdot\mathbf{c})+\beta(\mathbf{a}\cdot\mathbf{c})$

$\mathbf{a}\cdot(\beta\mathbf{b}+\gamma\mathbf{c}) \;=\; \beta(\mathbf{a}\cdot\mathbf{b})+\gamma(\mathbf{a}\cdot\mathbf{c})$

D. the norm of a vector

• the norm / magnitude / length of $\mathbf{a} = (a_1, a_2, a_3)$

$=_{dn}$ $|\mathbf{a}|$

$=_{rd}$ norm $\mathbf{a}$

$=_{df}$ $\sqrt{\mathbf{a} \cdot \mathbf{a}}$

$= \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

$\in$ nonneg scalar

• norm formation is a function

$|*| : V \to \mathbb{R}$

$\mathbf{a} \mapsto |\mathbf{a}|$

• to form the norm

square each component

& add

& take the nonegative square root

• another notation for the norm of a vector

denoted by a single bold - face letter is

the correspondng light - face letter eg $|\mathbf{a}| = a$ GG98 - 32

BP. norm

• $\mathbf{0}$ nullifies norm

$|\mathbf{0}| = 0$

• norm $\in$ pos def

$|\mathbf{a}| \geq 0$

$|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$

$|\mathbf{a}| > 0 \Leftrightarrow \mathbf{a} \neq \mathbf{0}$

• norm $\in$ even

$|-\mathbf{a}| = |\mathbf{a}|$

• norm $\in$ absolutely homogeneous

$|\alpha\mathbf{a}| = |\alpha||\mathbf{a}|$

• norm $\in$ subadditive

= norm satisfies triangle inequality

$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

• extended triangle inequalities

$$\left||\mathbf{a}| - |\mathbf{b}|\right| \le |\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$$

$$\left||\mathbf{a}| - |\mathbf{b}|\right| \le |\mathbf{a} - \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$$

$$|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$$

$$|\mathbf{a} + \mathbf{b} + \mathbf{c}| \le |\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}|$$

$$|\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}| \le |\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}|$$

etc

• the norm satisifies the parallelogram identity:
the sum of the squares
of the diagonals of a paralleogram
equals
the sum of the squares of the sides

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 \;=\; 2\left(|\mathbf{a}|^2 + |\mathbf{b}|^2\right)$$

• this is a form of the pythagorean theorem;
recognize it?

$$|\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$$

D. the cross product of two vectors

• the cross product of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$

$=_{dn}$ $\mathbf{a} \times \mathbf{b}$

$=_{rd}$ **a** cross **b**

$$=_{df} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =_{dn} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$= \left(a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1\right)$$

$\in$ vector

• vector / outer multiplication
is a binary operation in V

$$\times : V \times V \rightarrow V$$
$$(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \times \mathbf{b}$$

• to form the cross product
think of expanding
a third order determinant
along the first row

BP. cross product

• $\mathbf{0}$ factor nullifies cross product;
cross product vanishes if a factor vanishes

$$\mathbf{a} \times \mathbf{0} = \mathbf{0}$$

$$\mathbf{0} \times \mathbf{a} = \mathbf{0}$$

• cross product $\in$ anticommuative

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

• negation floats in a cross product

$$(-\mathbf{a}) \times \mathbf{b} = -\mathbf{a} \times \mathbf{b}$$

$$\mathbf{a} \times (-\mathbf{b}) = -\mathbf{a} \times \mathbf{b}$$

• double negation disappears in a cross product

$$(-\mathbf{a}) \times (-\mathbf{b}) = \mathbf{a} \times \mathbf{b}$$

• cross product $\in$ bihomogeneous

$$(\alpha \mathbf{a}) \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\beta \mathbf{b}) = \beta(\mathbf{a} \times \mathbf{b})$$

$$(\alpha \mathbf{a}) \times (\beta \mathbf{b}) = \alpha\beta(\mathbf{a} \times \mathbf{b})$$

• cross product $\in$ bilaterally distributive
over addition & subtraction

$$\mathbf{a} \times (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \times \mathbf{b} \pm \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} \pm \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} \pm \mathbf{b} \times \mathbf{c}$$

• cross product $\in$ bilinear ie linear in each factor

$$(\alpha \mathbf{a} + \beta \mathbf{b}) \times \mathbf{c} = \alpha(\mathbf{a} \times \mathbf{c}) + \beta(\mathbf{a} \times \mathbf{c})$$

$$\mathbf{a} \times (\beta \mathbf{b} + \gamma \mathbf{c}) = \beta(\mathbf{a} \times \mathbf{b}) + \gamma(\mathbf{a} \times \mathbf{c})$$

D. the determinant of three vectors

• the determinant of

$\mathbf{a} = (a_1, a_2, a_3),\ \mathbf{b} = (b_1, b_2, b_3),\ \mathbf{c} = (c_1, c_2, c_3)$

$=_{dn}\ |\mathbf{a}, \mathbf{b}, \mathbf{c}|$

$=_{rd}\ \det \mathbf{a}, \mathbf{b}, \mathbf{c}$

$$=_{df} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{(with vectors as rows)}$$

$$=_{df} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{(with vectors as columns)}$$

$\in$ scalar

• the determinant is a function

$$|*| : V \times V \times V \to \mathbb{R}$$

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) \mapsto |\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

• $|\mathbf{a},\mathbf{b},\mathbf{c}|$

$$= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}) = -(\mathbf{c}\times\mathbf{b})\cdot\mathbf{a}$$

$$= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) = -(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b}$$

$$= \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}) = -(\mathbf{b}\times\mathbf{a})\cdot\mathbf{c}$$

which represents the expansion
of the determinant
by cofactors / minors
along the three rows or the three columns
according to how the entries are written

• the duality of determinants wrt rows & columns
is reflected in the fact that **a**, **b**, **c**
may be consided equivalently
as rows or columns in the determinant $|\mathbf{a},\mathbf{b},\mathbf{c}|$

• here are combinatorial rules
for the expansion of a third - order determinant
into its $3! = 6$ terms;
perhaps expansion by diagonals is the easiest
but does not generalize to higher order determinants;
the determinant $|\mathbf{a}, \mathbf{b}, \mathbf{c}|$ equals
the sum of six products $\pm a_i b_j c_k$,
three pluses for even permutations of 1 2 3
&
three minuses for odd permutations of 1 2 3;
the latter rule is captured by the expression
$e_{ijk} a_i b_j c_k$
which uses
the permutation symbol e_{ijk}
&
the repeated index summation convention;
this generalizes to determinants of any order;
expansion by cofactors / minors along any row or column
also generalizes to determinants of any order

BP. determinant

• a **0** term nullifies the determinant

$$|\mathbf{a},\mathbf{b},\mathbf{0}| = 0$$

$$|\mathbf{a},\mathbf{0},\mathbf{b}| = 0$$

$$|\mathbf{0},\mathbf{a},\mathbf{b}| = 0$$

• two equal terms nullify the determinant

$$|\mathbf{a},\mathbf{a},\mathbf{b}| = 0$$

$$|\mathbf{a},\mathbf{b},\mathbf{a}| = 0$$

$$|\mathbf{b},\mathbf{a},\mathbf{a}| = 0$$

• negating a term negates the determinant

$$|-\mathbf{a},\mathbf{b},\mathbf{c}| = -|\mathbf{a},\mathbf{b},\mathbf{c}|$$

$$|\mathbf{a},-\mathbf{b},\mathbf{c}| = -|\mathbf{a},\mathbf{b},\mathbf{c}|$$

$$|\mathbf{a},\mathbf{b},-\mathbf{c}| = -|\mathbf{a},\mathbf{b},\mathbf{c}|$$

• the determinant $\in$ alternating

ie interchanging two terms

changes the sign of the determinant

$$|\mathbf{a}, \mathbf{c}, \mathbf{b}| = -|\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

$$|\mathbf{c}, \mathbf{b}, \mathbf{a}| = -|\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

$$|\mathbf{b}, \mathbf{a}, \mathbf{c}| = -|\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

also

a cyclic shift of terms

leaves the value of the determinant unchanged

$$|\mathbf{a}, \mathbf{b}, \mathbf{c}| = |\mathbf{b}, \mathbf{c}, \mathbf{a}| = |\mathbf{c}, \mathbf{a}, \mathbf{b}|$$

• the determinant $\in$ triadditive

ie additive in each term

$$|\mathbf{a}+\mathbf{a}',\mathbf{b},\mathbf{c}| = |\mathbf{a},\mathbf{b},\mathbf{c}| + |\mathbf{a}',\mathbf{b},\mathbf{c}|$$

$$|\mathbf{a},\mathbf{b}+\mathbf{b}',\mathbf{c}| = |\mathbf{a},\mathbf{b},\mathbf{c}| + |\mathbf{a},\mathbf{b}',\mathbf{c}|$$

$$|\mathbf{a},\mathbf{b},\mathbf{c}+\mathbf{c}'| = |\mathbf{a},\mathbf{b},\mathbf{c}| + |\mathbf{a},\mathbf{b},\mathbf{c}'|$$

• the determinant $\in$ trihomogeneous

ie homogeneous in each term

$$|\lambda\mathbf{a},\mathbf{b},\mathbf{c}| = \lambda|\mathbf{a},\mathbf{b},\mathbf{c}|$$

$$|\mathbf{a},\lambda\mathbf{b},\mathbf{c}| = \lambda|\mathbf{a},\mathbf{b},\mathbf{c}|$$

$$|\mathbf{a},\mathbf{b},\lambda\mathbf{c}| = \lambda|\mathbf{a},\mathbf{b},\mathbf{c}|$$

• the determinant $\in$ trilinear

ie linear in each term

$$|\alpha\mathbf{a}+\alpha'\mathbf{a}',\mathbf{b},\mathbf{c}| = \alpha|\mathbf{a},\mathbf{b},\mathbf{c}| + \alpha'|\mathbf{a}',\mathbf{b},\mathbf{c}|$$

$$|\mathbf{a},\beta\mathbf{b}+\beta'\mathbf{b}',\mathbf{c}| = \beta|\mathbf{a},\mathbf{b},\mathbf{c}| + \beta'|\mathbf{a},\mathbf{b}',\mathbf{c}|$$

$$|\mathbf{a},\mathbf{b},\gamma\mathbf{c}+\gamma'\mathbf{c}'| = \gamma|\mathbf{a},\mathbf{b},\mathbf{c}| + \gamma'|\mathbf{a},\mathbf{b},\mathbf{c}'|$$

• multiplying one term by a scalar
& adding to another term,
preserves the value of the determinant

$$|\mathbf{a} + \lambda\mathbf{b}, \mathbf{b}, \mathbf{c}| = |\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

$$|\mathbf{a} + \lambda\mathbf{c}, \mathbf{b}, \mathbf{c}| = |\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

$$|\mathbf{a}, \mathbf{b} + \lambda\mathbf{a}, \mathbf{c}| = |\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

$$|\mathbf{a}, \mathbf{b} + \lambda\mathbf{c}, \mathbf{c}| = |\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

$$|\mathbf{a}, \mathbf{b}, \mathbf{c} + \lambda\mathbf{a}| = |\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

$$|\mathbf{a}, \mathbf{b}, \mathbf{c} + \lambda\mathbf{b}| = |\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

• dot and cross interchange

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = |\mathbf{a}, \mathbf{b}, \mathbf{c}|$$

• the vanishing / nonvanishing of the determinant characterizes the linear dependence / independence of the terms

$$|\mathbf{a}, \mathbf{b}, \mathbf{c}| = 0 \Leftrightarrow (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \text{lin dep}$$

$$|\mathbf{a}, \mathbf{b}, \mathbf{c}| \neq 0 \Leftrightarrow (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \text{lin ind}$$

• rule for multiplying determinants

$$|\mathbf{a}, \mathbf{b}, \mathbf{c}| \cdot |\mathbf{d}, \mathbf{e}, \mathbf{f}| = \begin{vmatrix} \mathbf{a}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{e} & \mathbf{a}\cdot\mathbf{f} \\ \mathbf{b}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{e} & \mathbf{b}\cdot\mathbf{f} \\ \mathbf{c}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{e} & \mathbf{c}\cdot\mathbf{f} \end{vmatrix}$$

note the pattern:

think of the components of

a b c as written along the rows

&

think of the components of

d e f as written along the columns;

then take the dot product of

row into column

nine times;

this leads to the notion of

the product of two matrices

T. Cramer's Rule

as the solution of a vector equation

let

- $|\mathbf{a}, \mathbf{b}, \mathbf{c}| \neq 0$
- $x, y, z \in$ sca var

&

consider the vec eqn

(*) $\mathbf{a}x + \mathbf{b}y + \mathbf{c}z = \mathbf{d}$

then

- $\exists !$ sol of (*) viz

$$x = \frac{|\mathbf{d}, \mathbf{b}, \mathbf{c}|}{|\mathbf{a}, \mathbf{b}, \mathbf{c}|}$$

$$y = \frac{|\mathbf{a}, \mathbf{d}, \mathbf{c}|}{|\mathbf{a}, \mathbf{b}, \mathbf{c}|}$$

$$z = \frac{|\mathbf{a}, \mathbf{b}, \mathbf{d}|}{|\mathbf{a}, \mathbf{b}, \mathbf{c}|}$$

T. the solution of a system of vector equations

let

- $|\mathbf{a},\mathbf{b},\mathbf{c}| \neq 0$
- $\mathbf{x} \in$ vec var

&

consider the sys of vec eqns

$$(*)\begin{cases} \mathbf{a}\cdot\mathbf{x} = \alpha \\ \mathbf{b}\cdot\mathbf{x} = \beta \\ \mathbf{c}\cdot\mathbf{x} = \gamma \end{cases}$$

then

- $\exists !$ sol of (*) viz

$$\mathbf{x} = \frac{\alpha(\mathbf{b}\times\mathbf{c}) + \beta(\mathbf{c}\times\mathbf{a}) + \gamma(\mathbf{a}\times\mathbf{b})}{|\mathbf{a},\mathbf{b},\mathbf{c}|}$$

T. the solution of a system of vector equations

let

- $\mathbf{a} \neq \mathbf{0}$
- $\mathbf{x} \in$ vec var

&

consider the sys of vec eqns

$$(*)\begin{cases} \mathbf{a} \cdot \mathbf{x} = \alpha \\ \mathbf{a} \times \mathbf{x} = \mathbf{b} \end{cases}$$

then

- $\exists$ sol of (*) $\Leftrightarrow$ $\mathbf{a} \cdot \mathbf{b} = 0$
- $\mathbf{a} \cdot \mathbf{b} = 0$

$\Rightarrow$

$\exists!$ sol of (*) viz

$$\mathbf{x} = \frac{\mathbf{1}}{\mathbf{a}^2}(\alpha\mathbf{a} - \mathbf{a} \times \mathbf{b})$$

D. the three canonical basic vectors are:

$\mathbf{i} =_{df} (1,0,0) =_{cl}$ Isaac
$\mathbf{j} =_{df} (0,1,0) =_{cl}$ Jacob
$\mathbf{k} =_{df} (0,0,1) =_{cl}$ Kilroy

BP. canonical basic vectors

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0$$

$$\mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$$

$$\mathbf{k}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = 0$$

$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$$

$$\mathbf{a}\cdot\mathbf{i} = a_1$$

$$\mathbf{a}\cdot\mathbf{j} = a_2$$

$$\mathbf{a}\cdot\mathbf{k} = a_3$$

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{i})\mathbf{i} + (\mathbf{a}\cdot\mathbf{j})\mathbf{j} + (\mathbf{a}\cdot\mathbf{k})\mathbf{k}$$

$(\mathbf{i},\mathbf{j},\mathbf{k}) \in$ orthonormal basis of V

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}$$
$$\mathbf{j}\times\mathbf{k} = \mathbf{i}$$
$$\mathbf{k}\times\mathbf{i} = \mathbf{j}$$

$$\mathbf{j}\times\mathbf{i} = -\mathbf{k}$$
$$\mathbf{k}\times\mathbf{j} = -\mathbf{i}$$
$$\mathbf{i}\times\mathbf{k} = -\mathbf{j}$$

$$|\mathbf{i},\mathbf{j},\mathbf{k}| = 1$$

$$(\mathbf{i}\times\mathbf{a})\times\mathbf{i} = \mathbf{a}-(\mathbf{a}\cdot\mathbf{i})\mathbf{i}$$
$$(\mathbf{j}\times\mathbf{a})\times\mathbf{j} = \mathbf{a}-(\mathbf{a}\cdot\mathbf{j})\mathbf{j}$$
$$(\mathbf{k}\times\mathbf{a})\times\mathbf{k} = \mathbf{a}-(\mathbf{a}\cdot\mathbf{k})\mathbf{k}$$

$$\mathbf{a} = \frac{1}{2}\left[(\mathbf{i}\times\mathbf{a})\times\mathbf{i}+(\mathbf{j}\times\mathbf{a})\times\mathbf{j}+(\mathbf{k}\times\mathbf{a})\times\mathbf{k}\right]$$

• thinking of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as rows

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \mathrm{I}$$

&

thinking of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as columns

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix} = \mathrm{I}$$

wh

I = the 3×3 real identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

C. in the following we consider
basic correspondences between
the algebra of the vector space V
&
the geometry of the vector space V;
the combo of
algebraic formulas
&
geometric pictures
makes mighty mathematics;
that vector spaces have
both an algebraic persona and a geometric persona
is likely an important reason why
they are of great importance in mathematics;
the most likely primary reason for their importance
is that vector spaces crystallize the notion of
linear algebraic operations
viz
add, subtract, and multiply by a scalar

C. the fundamental connection between
the algebra & the geometry of vector space V
occurs in the correspondence between
a coordinate - triple of real numbers
= a vector as an ordered triple of real numbers
& a point of 3 - space, once a coordinate system
such as a rectangular coordinate system is chosen;
there is also a fundamental connection between
vectors as arrows
& the measure of certain physical quantities;
an arrow is characterized by
a length = a (positive) real number = a scalar
and
a direction;
there are many physical quantities
such as velocity and force whose measures
are also characterized by a scalar and a direction;
thus vectors spaces are
significantly useful in physical science;
there is also a notable appearance of vector spaces
in analysis which inp are highly pertinent to physics

□ midpoint of a line segment
& centroid of a triangle
& center of gravity $=_{ab}$ CG
of a rigid system of points
with equal masses at the points

- $\frac{1}{2}(\mathbf{a}+\mathbf{b})$

= the midpoint of the line segment
joining the points **a** and **b**
= CG of $\mathbf{a}, \mathbf{b}$

- $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$

= the centroid of the triangle
whose vertices are the points $\mathbf{a}, \mathbf{b}, \mathbf{c}$
= CG of $\mathbf{a}, \mathbf{b}, \mathbf{c}$

- $\frac{1}{n}(\mathbf{a}_1+\mathbf{a}_2+\cdots+\mathbf{a}_n)$ wh $n \in$ pos int

= CG of $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$

☐ formulas for the six basic trig fcns
of the angle between two vectors
let

- $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b}$

then

- $\sin\angle(\vec{\mathbf{a}},\vec{\mathbf{b}}) = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$
- $\csc\angle(\vec{\mathbf{a}},\vec{\mathbf{b}}) = \dfrac{|\mathbf{a}||\mathbf{b}|}{|\mathbf{a}\times\mathbf{b}|}$
- $\cos\angle(\vec{\mathbf{a}},\vec{\mathbf{b}}) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
- $\sec\angle(\vec{\mathbf{a}},\vec{\mathbf{b}}) = \dfrac{|\mathbf{a}||\mathbf{b}|}{\mathbf{a}\cdot\mathbf{b}}$
- $\tan\angle(\vec{\mathbf{a}},\vec{\mathbf{b}}) = \dfrac{|\mathbf{a}\times\mathbf{b}|}{\mathbf{a}\cdot\mathbf{b}}$
- $\cot\angle(\vec{\mathbf{a}},\vec{\mathbf{b}}) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$

D. orthogonal / perpendicular vectors

• **a** is orthogonal / perpendicular to **b**

= **a** and **b** are orthogonal / perpendicular

$=_{dn}$ $\mathbf{a} \perp \mathbf{b}$

$=_{rd}$ **a** perp **b**   wh perp ← perpendicular

$=_{df}$ $\mathbf{a} \cdot \mathbf{b} = 0$

R. characterizations
of orthogonal / perpendicular vectors
= the extended pythagorean theorem
tfsape

- **a** is orthogonal / perpendicular to **b**
- **a** and **b** are orthogonal / perpendicular
- $\mathbf{a} \perp \mathbf{b}$
- $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$
- $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$
- $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$
- $\mathbf{a} \cdot \mathbf{b} = 0$
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$
- $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b} \Rightarrow \vec{\mathbf{a}} \perp \vec{\mathbf{b}}$
- $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b} \Rightarrow \angle(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = \text{rt ang}$

these statements may be
interpreted geometrically
in the parallelogram with sides $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$

D. mergent vectors

• **a** is mergent with **b**

= **a** and **b** are mergent

$=_{dn}$ $\mathbf{a} \parallel \mathbf{b}$

$=_{rd}$ **a** merge **b**

$=_{df}$

$\exists \alpha, \beta . (\alpha \neq 0 \vee \beta \neq 0) \,\&\, \alpha \mathbf{a} = \beta \mathbf{b}$

wiet

$\exists \alpha . \alpha \mathbf{a} = \mathbf{b} \vee \exists \beta . \mathbf{a} = \beta \mathbf{b}$

• **a** is promergent with **b**

= **a** and **b** are promergent

$=_{dn}$ $\mathbf{a} \underset{+}{\parallel} \mathbf{b}$

$=_{rd}$ **a** pro **b**

$=_{df}$

$\exists \alpha, \beta . \alpha\beta > 0 \ \& \ \alpha \mathbf{a} = \beta \mathbf{b}$

wiet

$\exists \alpha . \alpha > 0 \ \& \ \alpha \mathbf{a} = \mathbf{b}$

wiet

$\exists \beta . \beta > 0 \ \& \mathbf{a} = \beta \mathbf{b}$

• **a** is antimergent with **b**

= **a** and **b** are antimergent

$=_{dn}$ $\mathbf{a} \underline{\|} \mathbf{b}$

$=_{rd}$ **a** anti **b**

$=_{df}$

$\exists\alpha,\beta.\alpha\beta < 0 \;\&\; \alpha\mathbf{a} = \beta\mathbf{b}$

wiet

$\exists\alpha.\alpha < 0 \;\&\; \alpha\mathbf{a} = \mathbf{b}$

wiet

$\exists\beta.\beta < 0 \;\&\mathbf{a} = \beta\mathbf{b}$

C. in regard to the manufactured terminology

mergent, promergent, antimergent

think of the arrows as merging

and pointing in the same or opposite directions

R. characterizations of mergence

tfsape

- **a** is mergent with **b**
- **a** and **b** are mergent
- $\mathbf{a} \parallel \mathbf{b}$
- $\exists \alpha, \beta . (\alpha \neq 0 \vee \beta \neq 0) \,\&\, \alpha \mathbf{a} = \beta \mathbf{b}$
- $\exists \alpha . \alpha \mathbf{a} = \mathbf{b} \vee \exists \beta . \mathbf{a} = \beta \mathbf{b}$
- $\mathbf{a} \times \mathbf{b} = \mathbf{0}$
- $|\mathbf{a} \times \mathbf{b}| = 0$
- $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$
- $(\mathbf{a}, \mathbf{b}) \in$ lin dep
- the points **a** and **b** lie on a line
  passing thru the origin
- the points $\mathbf{a}, \mathbf{b}, \mathbf{0}$ lie on a line
  in any order
- the points $\mathbf{a}, \mathbf{b}, \mathbf{0}$ are collinear
- $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b} \Rightarrow$ the arrows $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are collinear
- $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b} \Rightarrow$ the arrows $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have
  the same direction or opposite directions
- $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b} \Rightarrow \angle(\vec{\mathbf{a}}, \vec{\mathbf{b}}) =$ null ang or str ang

R. characterizations of promergence

let

- $\mathbf{a} \neq \mathbf{0}$ & $\mathbf{b} \neq \mathbf{0}$

then

tfsape

- **a** is promergent with **b**
- **a** and **b** are promergent
- $\mathbf{a} \underset{+}{\parallel} \mathbf{b}$
- $\exists \alpha, \beta . \alpha\beta > 0 \;\&\; \alpha \mathbf{a} = \beta \mathbf{b}$
- $\exists \alpha . \alpha > 0 \;\&\; \alpha \mathbf{a} = \mathbf{b}$
- $\exists \beta . \beta > 0 \;\&\; \mathbf{a} = \beta \mathbf{b}$
- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$
- the points **a** and **b** lie on a ray
  from the origin
- the points $\mathbf{a}, \mathbf{b}, \mathbf{0}$ lie on a line
  in the order $\mathbf{0}, \mathbf{a}, \mathbf{b}$ or $\mathbf{0}, \mathbf{b}, \mathbf{a}$
- the arrows $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have the same direction
- $\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}})$ = null ang

R. characterizations of antimergence

let

- $\mathbf{a} \neq \mathbf{0}$ & $\mathbf{b} \neq \mathbf{0}$

then

tfsape

- $\mathbf{a}$ is antimergent with $\mathbf{b}$
- $\mathbf{a}$ and $\mathbf{b}$ are antimergent
- $\mathbf{a} \underset{-}{\parallel} \mathbf{b}$
- $\exists \alpha, \beta . \alpha\beta < 0 \;\&\; \alpha\mathbf{a} = \beta\mathbf{b}$
- $\exists \alpha . \alpha < 0 \;\&\; \alpha\mathbf{a} = \mathbf{b}$
- $\exists \beta . \beta < 0 \;\&\; \mathbf{a} = \beta\mathbf{b}$
- $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}|$
- the points $\mathbf{a}, \mathbf{b}, \mathbf{0}$ lie on a line
  in the order $\mathbf{a}, \mathbf{0}, \mathbf{b}$
- the arrows $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have opposite directions
- $\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = \text{str ang}$

☐ distance formulas

• $|\mathbf{a}|$

= a

= distance of point **a** from origin

= length of arrow $\vec{\mathbf{a}}$ if $\mathbf{a} \neq \mathbf{0}$

• $|\mathbf{a} - \mathbf{b}|$

= distance between point **a** and point **b**

□ for the triangle with sides $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$

- $|\mathbf{a}|,\ |\mathbf{b}|,\ |\mathbf{a}-\mathbf{b}|\ =$ lengths of sides of triangle

- $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|\ =$ area of triangle

- $\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}-\mathbf{b}|}\ =$ altitude of triangle from origin

- $\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|}\ =$ altitude of triangle to side $\vec{\mathbf{a}}$

- $\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}\ =$ altitude of triangle to side $\vec{\mathbf{b}}$

□ for the parallelogram with sides $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$

- $|\mathbf{a}|$, $|\mathbf{b}|$ = lengths of sides of parallelogram

- $|\mathbf{a}+\mathbf{b}|$ = length of diagonal of parallelogram from origin

- $|\mathbf{a}-\mathbf{b}|$ = length of diagonal of parallelogram joining tips of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$

- $|\mathbf{a}\times\mathbf{b}|$ = area of parallelogram

□ for the parallelopiped with edges $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$

- $|\mathbf{a}+\mathbf{b}+\mathbf{c}|$

= length of space diagonal of parallelopiped
from origin

- $|\mathbf{a},\mathbf{b},\mathbf{c}|$

= signed volume of parallelopiped

☐ for the pyramid with edges $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$

- $\frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$

= area of base of pyramid opposite origin

- $\frac{1}{6}|\mathbf{a}, \mathbf{b}, \mathbf{c}|$

= signed volume of pyramid

- $\operatorname{sgn}|\mathbf{a}, \mathbf{b}, \mathbf{c}|$

= orientation of triad $(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}})$

☐ the three arrows $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ are coplanar

iff

the four points **a**, **b**, **c**, **0** are coplanar

iff

$|\mathbf{a},\mathbf{b},\mathbf{c}| = 0$

iff

$(\mathbf{a},\mathbf{b},\mathbf{c}) \in$ lin dep

☐ the four arrows $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, $\vec{\mathbf{d}}$ have coplanar tips

iff

the four points **a**, **b**, **c**, **d** are coplanar

iff

$|\mathbf{a}-\mathbf{b},\ \mathbf{b}-\mathbf{c},\ \mathbf{c}-\mathbf{d}| = 0$

(this determinant has
many different but equal forms)

□ $\mathbf{c} = \mathbf{a} \times \mathbf{b} \neq \mathbf{0}$

$\Rightarrow$

$\vec{\mathbf{c}} \perp \vec{\mathbf{a}}$ & $\vec{\mathbf{c}} \perp \vec{\mathbf{b}}$

□ $\mathbf{c} = \mathbf{a} \times \mathbf{b} \neq \mathbf{0}$

$\Rightarrow$

$(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}) \in$ triad

with the same orientation as

the basic triad $(\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}})$

□ the three points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are collinear

iff

$\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = \mathbf{0}$

iff

$(\mathbf{a} - \mathbf{b}) \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$

(and iff cyclically)

☐ first triangle inequality

$|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$

& = holds

iff $\mathbf{a}=\mathbf{0} \vee \mathbf{b}=\mathbf{0} \vee \vec{\mathbf{a}} \parallel_{+} \vec{\mathbf{b}}$

☐ second triangle inequality

$|\mathbf{a}-\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$

& = holds

iff $\mathbf{a}=\mathbf{0} \vee \mathbf{b}=\mathbf{0} \vee \vec{\mathbf{a}} \parallel_{-} \vec{\mathbf{b}}$

☐ third triangle inequality

$\big||\mathbf{a}| - |\mathbf{b}|\big| \le |\mathbf{a}+\mathbf{b}|$

& = holds

iff $\mathbf{a}=\mathbf{0} \vee \mathbf{b}=\mathbf{0} \vee \vec{\mathbf{a}} \parallel_{-} \vec{\mathbf{b}}$

☐ fourth triangle inequality

$\big||\mathbf{a}| - |\mathbf{b}|\big| \le |\mathbf{a}-\mathbf{b}|$

& = holds

iff $\mathbf{a}=\mathbf{0} \vee \mathbf{b}=\mathbf{0} \vee \vec{\mathbf{a}} \parallel_{+} \vec{\mathbf{b}}$

☐ unified triangle inequality

$$\left||\mathbf{a}| - |\mathbf{b}|\right| \leq |\mathbf{a} \pm \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

☐ generalized triangle inequality

$$\left|\sum_{i=1}^{n} \mathbf{a_i}\right| \leq \sum_{i=1}^{n} |\mathbf{a_i}|$$

wh n $\in$ pos int

□ Cauchy - Schwarz inequality

$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$

& = holds

iff $\mathbf{a} = \mathbf{0} \vee \mathbf{b} = \mathbf{0} \vee \vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$

□ Lagrange inequality

$|\mathbf{a} \times \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$

& = holds

iff $\mathbf{a} = \mathbf{0} \vee \mathbf{b} = \mathbf{0} \vee \vec{\mathbf{a}} \perp \vec{\mathbf{b}}$

□ Hadamard inequality

$\left| |\mathbf{a}, \mathbf{b}, \mathbf{c}| \right| \leq |\mathbf{a}| \cdot |\mathbf{b}| \cdot |\mathbf{c}|$

& = holds

iff $\mathbf{a} = \mathbf{0} \vee \mathbf{b} = \mathbf{0} \vee \mathbf{c} = \mathbf{0} \vee (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \perp$

□ vector equations of
lines, planes, curves, surfaces

• t, u, v, x, y, z ∈ real nr var

• $\mathbf{r} = (x, y, z) \in$ real vec var
$=_{cl}$ the running point (in an older terminology)

• the line parallel to $\vec{\mathbf{a}}$ wh $\mathbf{a} \neq \mathbf{0}$
& passing thru the point $\mathbf{b}$
has parametric vector equation
$\mathbf{r} = \mathbf{a}t + \mathbf{b} \quad (-\infty < t < \infty)$

• the plane parallel to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ wh $(\mathbf{a}, \mathbf{b}) \in$ lin ind
& ∴ with normal direction $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$
& passing thru the point $\mathbf{c}$
has parametric vector equation
$\mathbf{r} = \mathbf{a}u + \mathbf{b}v + \mathbf{c} \quad (-\infty < u, v < \infty)$

• a (skew) curve
has parametric vector equation
of the form

$$\mathbf{r} = \mathbf{a}f(t) + \mathbf{b}g(t) + \mathbf{c}h(t)$$

wh $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in$ lin dep

& $f, g, h \in$ real-val say C^1 on int of $\mathbb{R}$

• a (skew) curve
has parametric vector equation
of the form

$$\mathbf{r} = \mathbf{i}f(t) + \mathbf{j}g(t) + \mathbf{k}h(t)$$

& $f, g, h \in$ real-val say C^1 on int of $\mathbb{R}$

• the circular helix
has parametric vector equation

$$\mathbf{r} = \mathbf{i}\cos t + \mathbf{j}\sin t + \mathbf{k}\,t \quad (-\infty < t < \infty)$$

• a surface

has parametric vector equation

of the form

$\mathbf{r} = \mathbf{a}f(u,v) + \mathbf{b}g(u,v) + \mathbf{c}h(u,v)$

wh $(\mathbf{a},\mathbf{b},\mathbf{c}) \in$ lin dep

& $f,g,h \in$ real - val say C^2 on reg of $\mathbb{R}^2$

• a surface

has parametric vector equation

of the form

$\mathbf{r} = \mathbf{i}f(u,v) + jg(u,v) + \mathbf{k}h(u,v)$

wh $f,g,h \in$ real - val say C^2 on reg of $\mathbb{R}^2$

• the hyperbolic paraboloid

has parametric vector equation

$\mathbf{r} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}uv \quad (-\infty < u,v < \infty)$

• the line with direction $\vec{\mathbf{d}}$ wh $\mathbf{d} \neq \mathbf{0}$

& passing thru the point $\mathbf{a}$

has vector equation

$\mathbf{d} \times (\mathbf{r} - \mathbf{a}) = \mathbf{0}$

• the line passing

thru the distinct points $\mathbf{a}$ and $\mathbf{b}$

has vector equation

$(\mathbf{a} - \mathbf{b}) \times (\mathbf{r} - \mathbf{a}) = \mathbf{0}$

wiet

$(\mathbf{a} - \mathbf{b}) \times (\mathbf{r} - \mathbf{b}) = \mathbf{0}$

wiet

$(\mathbf{a} - \mathbf{b}) \times \mathbf{r} = \mathbf{a} \times \mathbf{b}$

• the plane with normal direction $\vec{\mathbf{n}}$ wh $\mathbf{n} \neq \mathbf{0}$
& passing thru the point $\mathbf{a}$
has vector equation
$\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) \;=\; 0$

• the plane passing thru
the three noncollinear points $\mathbf{a}, \mathbf{b}, \mathbf{c}$
has vector equation
$|\mathbf{r} - \mathbf{a},\ \mathbf{a} - \mathbf{b},\ \mathbf{a} - \mathbf{c}| = 0$
(& many other equivalent similar forms)
wiet
$(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) \cdot \mathbf{r} = |\mathbf{a}, \mathbf{b}, \mathbf{c}|$
wiet
$|\mathbf{r}, \mathbf{a}, \mathbf{b}| + |\mathbf{r}, \mathbf{b}, \mathbf{c}| + |\mathbf{r}, \mathbf{c}, \mathbf{a}| = |\mathbf{a}, \mathbf{b}, \mathbf{c}|$

• the unit sphere
has vector equation
$|\mathbf{r}| = 1$

• the sphere with center **a**
& radius ρ
has vector equation
$|\mathbf{r} - \mathbf{a}| = \rho$

• the prolate ellipsoid of revolution
with foci at distinct points **a** and **b**
has vector equation
$|\mathbf{r} - \mathbf{a}| + |\mathbf{r} - \mathbf{b}| = \rho$ wh $\rho \in$ real nr $> |\mathbf{a} - \mathbf{b}|$

☐ some distance formulas

• the distance from the point **p**
to the line with parametric vector equation
$\mathbf{r} = \mathbf{d}t + \mathbf{a} \quad (-\infty < t < \infty)$
is

$$\frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$$

$$= \frac{|\mathbf{p} \times \mathbf{d} + \mathbf{d} \times \mathbf{a}|}{|\mathbf{d}|}$$

• the distance from the point **p**
to the line thru
the distinct points **a** and **b**
is

$$\frac{|(\mathbf{p} - \mathbf{a}) \times (\mathbf{a} - \mathbf{b})|}{|\mathbf{a} - \mathbf{b}|}$$

$$= \frac{|(\mathbf{p} - \mathbf{b}) \times (\mathbf{a} - \mathbf{b})|}{|\mathbf{a} - \mathbf{b}|}$$

$$= \frac{|\mathbf{p} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{p}|}{|\mathbf{a} - \mathbf{b}|}$$

• the distance from the point **p**

to the plane with vector equation

$$\mathbf{n} \cdot \mathbf{r} = \alpha$$

is

$$\frac{|\mathbf{n} \cdot \mathbf{p} - \alpha|}{|\mathbf{n}|}$$

□ a few more algebraic vector identities

• double cross identities

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \;=\; (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;=\; (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$

• Jacobi identities

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} \;=\; \mathbf{0}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \;=\; \mathbf{0}$$

• Lagrange identity

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) \;=\; \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

- $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}-\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$

- $\big((\mathbf{a}\times\mathbf{b})\times\mathbf{c}\big)\cdot\big(\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\big)=(\mathbf{a}\cdot\mathbf{c})\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}$

☐ ¿when does the cross product ...?

• ¿when does the cross product idempote?

$$\mathbf{a} \times \mathbf{a} = \mathbf{a}$$

$$\Leftrightarrow$$

$$\mathbf{a} = \mathbf{0}$$

• ¿when does the cross product commute?

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$$

$$\Leftrightarrow$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

• ¿when does the cross product associate?

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

$$\Leftrightarrow$$

$$(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

C. matrices & linear transformations & analysis
come next;
there are algebraic vector spaces of dimension n
where n is any nonnegative integer
& over any field or division ring
instead of just the real number field ℝ;
the vectors spaces of infinite dimension
are generally called
linear spaces,
and over the real field ℝ or the complex field ℂ,
occur in analysis

C. we have taken arrows based at the origin
to canonically correspond to
ordered real number triples;
the correspondence could also be set up
based on any other point;
indeed the correspondence could be between
a number triple & a flight of arrows
viz
two directed line segments are equivalent
iff
they are the similarly directed sides of a parallelogram;
this notion is often useful in physical applications