Cramer's Rule

#92 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2003

GG92-1 (19)

500 Angell St #414
Providence RI 02906

GG92-2

□ Cramer' s Rule
concerns the solution
of a system of n linear equations in n unknowns
wh $n \in$ pos int
by the use of determinants;
the general case of Cramer' s Rule
for n linear equations in n unknowns
is notationally rather fussy to state in symbols;
instead we here state Cramer' s Rule in words
for the general case
and then state Cramer' s Rule explicitly in symbols
for the first three cases
$n = 1, 2, 3$
so that the symbolic rendering of the general case
becomes obvious;
considerations may be thought of
as taking place in any field F
eg
the rational field $\mathbb{Q}$ or the real field $\mathbb{R}$ or the complex field $\mathbb{C}$

• The Big Idea of Cramer's Rule

here is an informal verbal description
of the algorithm called Cramer's Rule;
start with a system of n linear equations in n unknowns,
the equations being written horizontally
and stacked vertically,
where the left hand sides of the equations
have the unknowns
appearing in the same order from left to right
& the constant terms
are on the right hand sides of the equations;
form the $n \times n$ 'denominator' determinant D
on the coefficients of the unknowns;
for each unknown
form a 'numerator' $n \times n$ determinant from D
by replacing
the column of the coefficients of this unknown
by the column of constant terms;
assume that $D \neq 0$;

then

the system has a unique solution

where the value of each unknown

is

the quotient of

the corresponding 'numerator' determinant

by the 'denominator' determinant

(1) Cramer's Rule

for one linear equation in one unknown

consider the system

$* \{ax = b$

of one linear equation in one unknown

x

&

form the 'denominator' determinant

$D = |a| = a$

on the coefficient of the unknown

&

form the 'x – numerator' determinant

$D_x = |b| = b$

from D by replacing

the column of the coefficient of x

with the column of the constant term

then

$D \neq 0$

$\Rightarrow$

$\exists !$ sol of * viz

$$x = \frac{D_x}{D}$$

(2) Cramer's Rule

for two linear equations in two unknowns

consider the system

$$* \begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

of two linear equations in two unknowns

x, y

&

form the 'denominator' determinant

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

on the coefficients of the unknowns

&

form the ' x – numerator' determinant

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

from D by replacing

the column of the coefficients of x

with the column of the constant terms

&

form the ' y – numerator' determinant

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

from D by replacing

the column of the coefficients of y

with the column of the constant terms

then

$D \neq 0$

$\Rightarrow$

$\exists !$ sol of * viz

$$\begin{cases} x = \dfrac{D_x}{D} \\ y = \dfrac{D_y}{D} \end{cases}$$

(3) Cramer's Rule

for three linear equations in three unknowns

consider the system

$$* \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

of three linear equations in three unknowns

x, y, z

&

form the 'denominator' determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

on the coefficients of the unknowns

&

form the ' x – numerator' determinant

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

from D by replacing
the column of the coefficients of x
with the column of the constant terms

&

form the ' y – numerator' determinant

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

from D by replacing
the column of the coefficients of y
with the column of the constant terms

&

form the ' z – numerator' determinant

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

from D by replacing
the column of the coefficients of z
with the column of the constant terms

then

$D \neq 0$

$\Rightarrow$

$\exists!$ sol of * viz

$$\begin{cases} x = \dfrac{D_x}{D} \\ y = \dfrac{D_y}{D} \\ z = \dfrac{D_z}{D} \end{cases}$$

□ a sketch of the proof of Cramer's Rule

Cramer's Rule may be proved
efficiently & insightfully
by using the properties of determinants;
in order to reduce the quantity of notation
consider the case for $n = 2$,
the general case evidently following the same ideas;
let

- $F \in$ field
- $a_1, b_1, c_1, a_2, b_2, c_2, x, y \in F$
- define D, D_x, D_y to be
  the 2×2 determinants as above
- assume $D \neq 0$

then

• if

$$x = \frac{D_x}{D} \ \& \ y = \frac{D_y}{D},$$

then

$$a_1x + b_1y = c_1$$

$$\Leftrightarrow$$

$$a_1D_x + b_1D_y = c_1D$$

$$\Leftrightarrow$$

$$a_1D_x + b_1D_y - c_1D = 0$$

which is true

because the LHS is the negative of

the expansion along the first row

of a 3×3 determinant with two rows identical

& thus

$$a_1x + b_1y = c_1$$

& similarly

$$a_2x + b_2y = c_2$$

• if

$a_1x + b_1y = c_1$ & $a_2x + b_2y = c_2$,

then

$$xD = x\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 \\ a_2x & b_2 \end{vmatrix} = \begin{vmatrix} a_1x + b_1y & b_1 \\ a_2x + b_{2y} & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = D_x$$

& thus

$xD = D_x$

& similarly

$yD = D_y$

whence

$$x = \frac{D_x}{D} \quad \& \quad y = \frac{D_y}{D}$$

QED

☐ bioline

Gabriel Cramer

1704 - 1752

Swiss

algebraist, analyst, editor, expositor;

published Cramer's Rule in 1750;

Cramer's Rule was previously known to Maclaurin

probably by 1729

& was published posthumously in 1748