Good Things about the Gudermannian

#88 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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GG88-1 (31)

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GG88-2

□ $x, y, u, t \in$ real nr var

□ the real gudermannian function

$=_{ab}$ the gudermannian

$=_{pr}$ goohd - er - MAHN - ee - en

$=_{dn}$ $\mathrm{gd}\, x \quad (-\infty < x < \infty)$

wh gd ← gudermannian

$=_{rd}$ goohd - er x

$=_{df}$ $\mathrm{Tan}^{-1} \sinh x$

□ the real gudermannian

constitutes a real bridge between

the real trigonometric functions

and

the real hyperbolic functions

viz

$y = \operatorname{gd} x$

$\Rightarrow$

$\sin y = \tanh x$

$\cos y = \operatorname{sech} x$

$\tan y = \sinh x$

$\cot y = \operatorname{csch} x$

$\sec y = \cosh x$

$\csc y = \coth x$

&

$\tan \frac{y}{2} = \tanh \frac{x}{2}$

$\cot \frac{y}{2} = \coth \frac{x}{2}$

□ the correspondence between
the trig fcns and the hyp fcns
via the gudermannian
produces a correspondence between
trig identities and hyp identities;
eg looking at
the three trig pythagorean identities

trig: $\sin^2 x + \cos^2 x = 1$

hyp: $\tanh^2 x + \operatorname{sech}^2 x = 1$

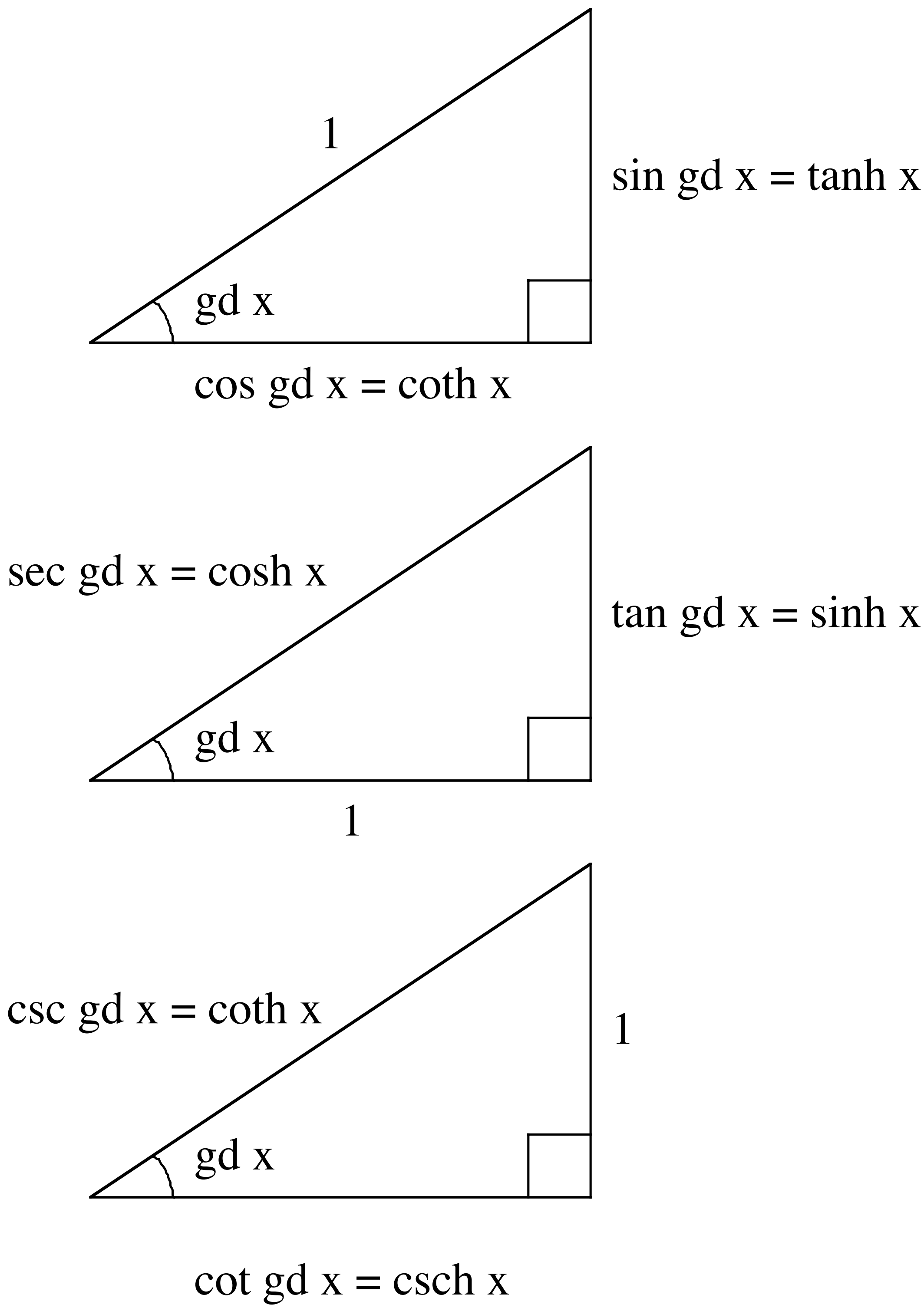
trig: $1 + \tan^2 x = \sec^2 x$

hyp: $1 + \sinh^2 x = \cosh^2 x$

trig: $1 + \cot^2 x = \csc^2 x$

hyp: $1 + \operatorname{csch}^2 x = \coth^2 x$

□ the corespondence
between trig fcns & hyp fcns
via the gudermannian
has a geometric description
which is given by the following
three labeled right triangles


1
sin gd x = tanh x
gd x
cos gd x = coth x
sec gd x = cosh x
tan gd x = sinh x
gd x
1
csc gd x = coth x
1
gd x
cot gd x = csch x

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☐ forms of the gudermannian

$$\begin{aligned}
&\operatorname{gd} x \\
&= \sin^{-1} \tanh x \\
&= \cos^{-1} \operatorname{sech} x \\
&= \tan^{-1} \sinh x \\
&= \cot^{-1} \operatorname{csch} x \\
&= \sec^{-1} \cosh x \\
&= \csc^{-1} \coth x \\
&= 2\tan^{-1} \tanh \frac{x}{2} \\
&= 2\tan^{-1} e^{x} - \frac{\pi}{2} \\
&= \int_0^x \operatorname{sech} t \, dt
\end{aligned}$$

(domains & ranges have to be specified)

GG88-8

□ forms of the inverse gudermannian

$$\begin{aligned}
&\operatorname{gd}^{-1} x \\
&= \sinh^{-1} \tan x \\
&= \cosh^{-1} \sec x \\
&= \tanh^{-1} \sin x \\
&= \coth^{-1} \csc x \\
&= \operatorname{sech}^{-1} \cos x \\
&= \operatorname{csch}^{-1} \cot x \\
&= 2 \tanh^{-1} \tan \frac{x}{2} \\
&= \log \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \\
&= \log (\sec x + \tan x) \\
&= \int_0^x \sec t \, dt
\end{aligned}$$

(domains & ranges have to be specified)

GG88-9

□ properties of the gudermannian $y = \operatorname{gd} x$
and its graph

Δ the function $y = \operatorname{gd} x$ has these properties

- domain: $-\infty < x < \infty$
- range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$
- class: analytic
- parity: odd
- strictly increasing
- $\operatorname{gd}(0) = 0$
- $\operatorname{gd} x > 0 \iff x > 0$
- $\operatorname{gd} x < 0 \iff x < 0$
- $\exists \lim_{x \to \infty} \operatorname{gd} x = \frac{\pi}{2}$
- $\exists \lim_{x \to -\infty} \operatorname{gd} x = -\frac{\pi}{2}$

Δ the graph of $y = \text{gd}\, x$ has these properties

- thru origin with slope 1
- symmetric wrt origin
- steadily rising
- asymptotic to horizontal line $y = \frac{\pi}{2}$
- asymptotic to horizontal line $y = -\frac{\pi}{2}$
- flex point at origin
- concave down for $x > 0$
- concave up for $x < 0$

• parametric equations, first form

$$\begin{cases} x = 2\tanh^{-1} t \\ y = 2\tan^{-1} t \end{cases}$$

wh $|t| < 1$

• parametric equations, second form

$$\begin{cases} x = 2\coth^{-1} t \\ y = 2\cot^{-1} t \end{cases}$$

wh $|t| > 1$

note: the values of the inv trig fcns

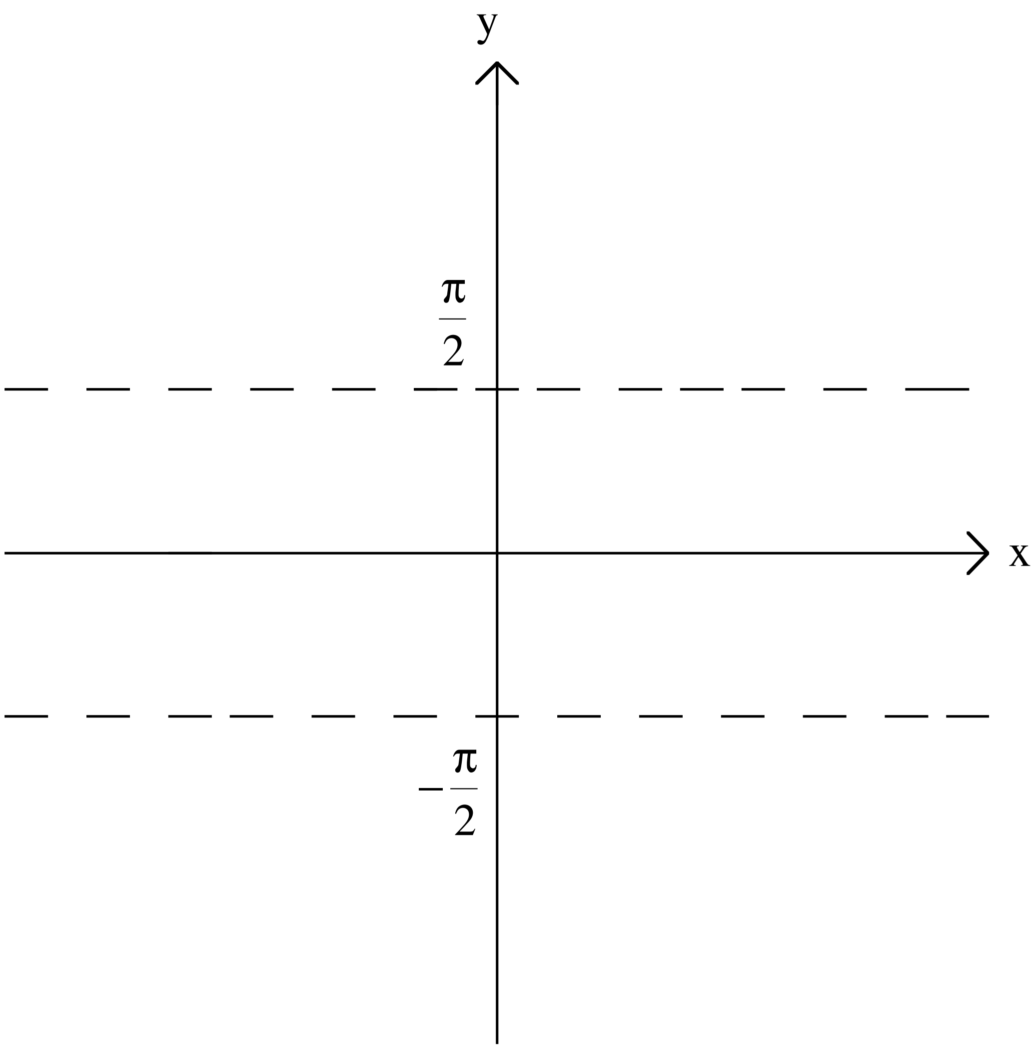
$\tan^{-1} t$ & $\cot^{-1} t$

are to be properly chosen

☐ do - it - yourself sketch:

graph of the gudermannian

$y = \text{gd}\, x \quad (-\infty < x < \infty)$



□ properties of the inverse gudermannian $y = gd^{-1} x$ and its graph

Δ the function $y = gd^{-1} x$ has these properties

- domain: $-\frac{\pi}{2} < x < \frac{\pi}{2}$
- range: $-\infty < y < \infty$
- class: analytic
- parity: odd
- strictly increasing
- $gd^{-1}(0) = 0$
- $gd^{-1} x > 0 \iff x > 0$
- $gd^{-1} x < 0 \iff x < 0$
- $gd^{-1} x \to \infty$ as $x \uparrow \frac{\pi}{2}$
- $gd^{-1} x \to -\infty$ as $x \downarrow -\frac{\pi}{2}$

Δ the graph of $y = gd^{-1}x$ has these properties

- thru origin with slope 1
- symmetric wrt origin
- steadily rising
- asymptotic to vertical line $x = \frac{\pi}{2}$
- asymptotic to vertical line $x = -\frac{\pi}{2}$
- flex point at origin
- concave up for $x > 0$
- concave down for $x < 0$

• parametric equations, first form

$$\begin{cases} x = 2\tan^{-1} t \\ y = 2\tanh^{-1} t \end{cases}$$

wh $|t| < 1$

• parametric equations, second form

$$\begin{cases} x = 2\cot^{-1} t \\ y = 2\coth^{-1} t \end{cases}$$

wh $|t| > 1$

note: the values of the inv trig fcns

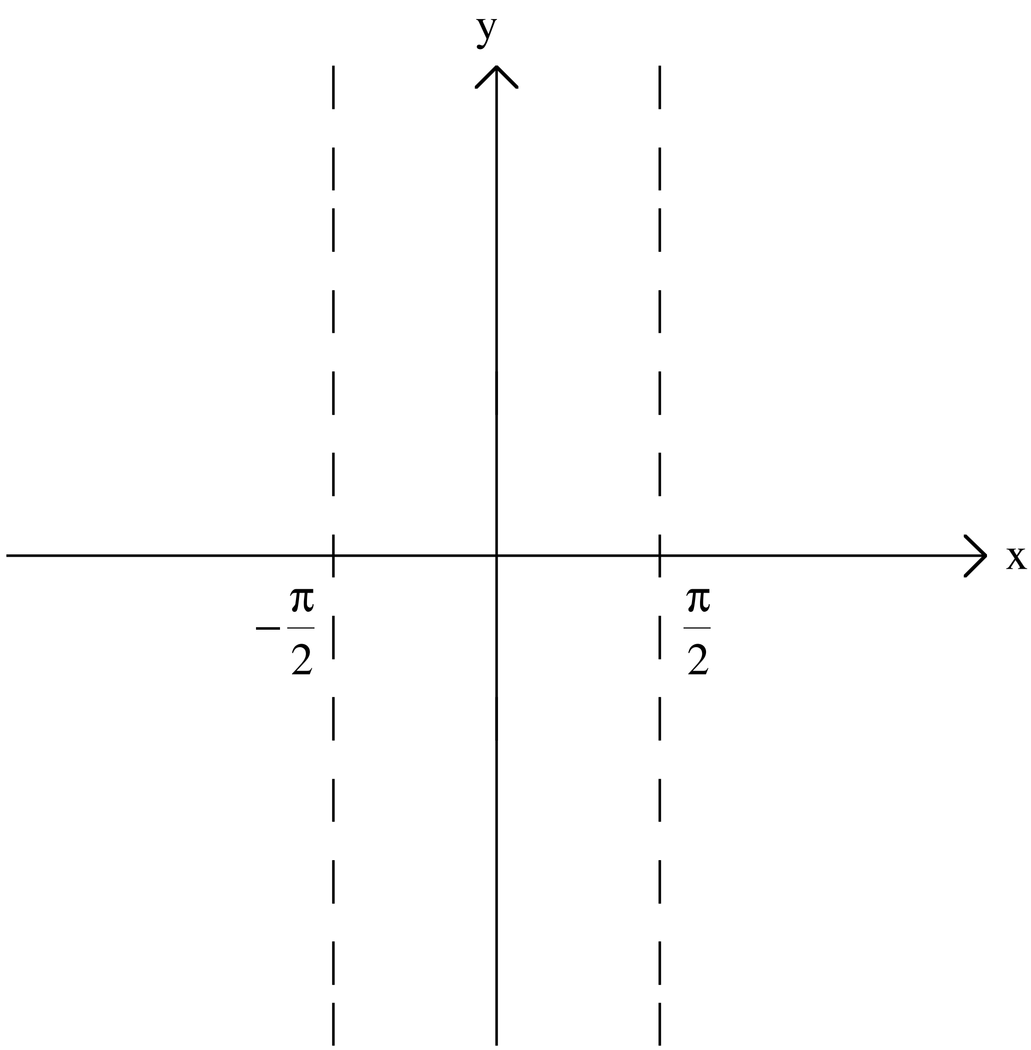
$\tan^{-1} t$ & $\cot^{-1} t$

are to be properly chosen

☐ do - it - yourself sketch:

graph of the inverse gudermannian

$$y = \mathrm{gd}^{-1}x \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$



□ derivatives and differentials

- $\frac{d}{dx} \operatorname{gd} x = \operatorname{sech} x$

- $\frac{d}{dx} \operatorname{gd}^{-1} x = \sec x$

- $d \operatorname{gd} x = \operatorname{sech} x \, dx$

- $d \operatorname{gd}^{-1} x = \sec x \, dx$

□ indefinite and definite integrals

- $\int \operatorname{sech} x \, dx = \operatorname{gd} x + C$

- $\int \sec x \, dx = \operatorname{gd}^{-1} x + C$

- $\int_0^x \operatorname{sech} t \, dt = \operatorname{gd} x$

- $\int_0^x \sec t \, dt = \operatorname{gd}^{-1} x$

□ the Maclaurin series for $\operatorname{gd} x$

$$\operatorname{gd} x$$

$$= x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{61x^7}{5040} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n+1)!} x^{2n+1}$$

$$\text{IC}: -1 < x < 1$$

□ the Maclaurin series for $gd^{-1} x$

$$gd^{-1} x$$

$$= x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n+1)!} x^{2n+1}$$

$$IC : -\frac{\pi}{2} < x < \frac{\pi}{2}$$

□ more formulas

- $\tanh \frac{1}{2} x = \tan \frac{1}{2} \operatorname{gd} x$ & $\coth \frac{1}{2} x = \cot \frac{1}{2} \operatorname{gd} x$

- e^x

$= \sec \operatorname{gd} x + \tan \operatorname{gd} x$

$= \dfrac{1 + \sin \operatorname{gd} x}{\cos \operatorname{gd} x}$

$= \dfrac{\cos \operatorname{gd} x}{1 - \sin \operatorname{gd} x}$

$= \tan\left(\frac{1}{2} \operatorname{gd} x + \frac{\pi}{4}\right)$

- $\operatorname{gd}^{-1}\left(x + \frac{\pi}{2}\right) = \log(\csc x - \cot x)$

- $\dfrac{d}{dx} \operatorname{gd}^{-1}\left(x + \frac{\pi}{2}\right) = \csc x$

- $\int \csc x \, dx = \operatorname{gd}^{-1}\left(x + \frac{\pi}{2}\right) + C$

□ the six basic trig functions
may be rationalized
by the substitution

$$u = \tan\frac{x}{2}$$

viz

$$\sin x = \frac{2u}{1+u^2} \qquad \cos x = \frac{1-u^2}{1+u^2}$$

$$\tan x = \frac{2u}{1-u^2} \qquad \cot x = \frac{1-u^2}{2u}$$

$$\sec x = \frac{1+u^2}{1-u^2} \qquad \csc x = \frac{1+u^2}{2u}$$

&

$$dx = \frac{2\,du}{1+u^2}$$

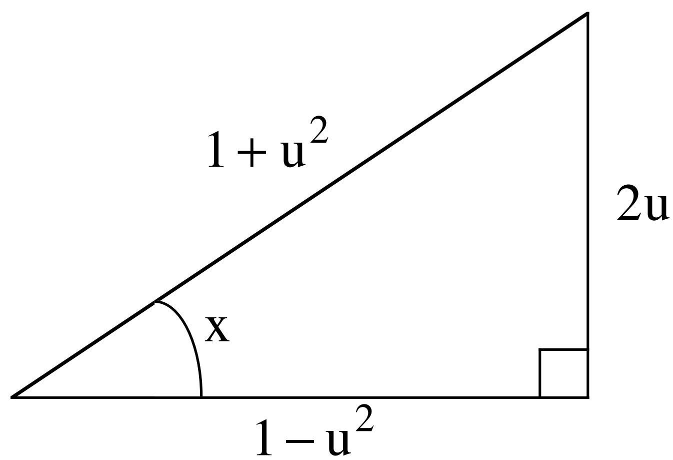
so that some trig integrands
may be rationalized by this substitution

- a geometric description

of the substitution

$$u = \tan\frac{x}{2}$$

is given by the labeled right triangle



□ the six basic hyp functions
may be rationalized
by the substitution

$$u = \tanh \frac{x}{2}$$

viz

$$\sinh x = \frac{2u}{1-u^2} \qquad \cosh x = \frac{1+u^2}{1-u^2}$$

$$\tanh x = \frac{2u}{1+u^2} \qquad \coth x = \frac{1+u^2}{2u}$$

$$\operatorname{sech} x = \frac{1-u^2}{1+u^2} \qquad \operatorname{csch} x = \frac{1-u^2}{2u}$$

&

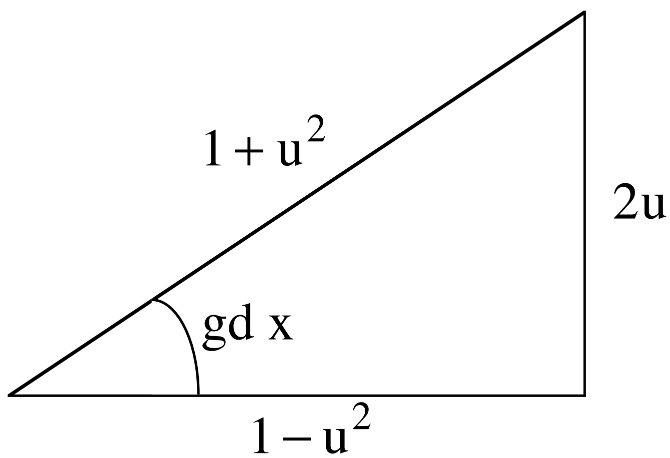
$$dx = \frac{2\,du}{1-u^2}$$

so that some hyp integrands
may be rationalized by this substitution

• a geometric description of the substitution

$$u = \tanh\frac{x}{2}$$

is given by the labeled right triangle



□ a definite integral

the area of
the region in QI bounded by
the curve $y = \text{gd}\, x$
& the horizontal line $y = \frac{\pi}{2}$
& the y – axis
=
the area of
the region in QI bounded by
the curve $y = \text{gd}^{-1} x$
& the vertical line $x = \frac{\pi}{2}$
& the x – axis

$$= \int_0^\infty \left(\frac{\pi}{2} - \text{gd}\, x\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \text{gd}^{-1} x \, dx$$

$$= 2\int_0^{\frac{\pi}{4}} \log\cot x \, dx$$

$$= 2¢$$

$$\approx 1.83+$$

wh

¢

$$=_{\text{cl}} \text{Catalan's constant}$$

$$=_{\text{df}} 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^2}$$

$$= 0.91596\ 55941\ 77219\ 01505\ 46035+$$

□ further uses of the gudermannian
pop up in various places such as
the theory of elliptic functions,
noneuclidean geometry,
physics of the pendulum,
and cartography;
indeed in the Mercator map projection
the vertical distance from the equator
of a location on the chart
is given by $\mathrm{gd}^{-1}\vartheta$
where ϑ is the latitude of the location

□ bioline

Christoph Gudermann

1798 - 1852

German

analyst, geometer; teacher of Weierstrass;
name ' gudermannian' and notation ' gd'
in present usage were introduced by Cayley
in honor of Gudermann' s work in the area

□ IMHO

the gudermannian should appear
in several exercises scattered thruout
the undergraduate calculus courses;
for example, it helps to clarify that mysterious formula
for the indefinite integral of the secant
and indeed provides a short formula for it;
likewise for the cosecant;
it correlates the trig functions
and the hyperbolic functions
in a pleasant and surprising manner