Series Expansions for the Twelve Basic Real Trigonometric and Hyperbolic Functions and Their Inverses

#87 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2003

GG87-1 (29)

500 Angell St #414
Providence RI 02906

GG87-2

- $\sin x$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

IC: $-\infty < x < \infty$

- cos x

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

IC: $-\infty < x < \infty$

- $\tan x$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n}\left(2^{2n}-1\right)B_{2n}}{(2n)!} x^{2n-1}$$

IC: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

- cot x

$$= \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \frac{1}{4725}x^7 - \cdots$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1}$$

$\text{IC}: \; 0 < |x| < \pi$

- $\sec x$

$$= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n}$$

$$\text{IC}: \; -\frac{\pi}{2} < x < \frac{\pi}{2}$$

- $\csc x$

$$= \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \frac{127}{604800}x^7 + \cdots$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(2^{2n} - 2\right)B_{2n}}{(2n)!} x^{2n-1}$$

$$\text{IC}:\ 0 < |x| < \pi$$

- $\mathrm{Sin}^{-1}x$

$$= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots$$

$$= x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots$$

$$= x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot \;\cdots\cdot(2n-1)}{2\cdot 4\cdot\;\cdots\;\cdot(2n)(2n+1)}x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1}$$

IC: $-1 \le x \le 1$

- $\text{Cos}^{-1} x$

$$= \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 - \frac{5}{112}x^7 - \cdots$$

$$= \frac{\pi}{2} - x - \frac{1}{2\cdot 3}x^3 - \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 - \cdots$$

$$= \frac{\pi}{2} - x - \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot \ \cdots\cdot(2n-1)}{2\cdot 4\cdot \ \cdots\ \cdot(2n)(2n+1)}x^{2n+1}$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1}$$

IC: $-1 \le x \le 1$

- $\operatorname{Tan}^{-1} x$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

IC: $-1 \le x \le 1$

- $\operatorname{Tan}^{-1} x$

$$= \frac{\pi}{2}\operatorname{sgn} x - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \cdots$$

$$= \frac{\pi}{2}\operatorname{sgn} x + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)x^{2n+1}}$$

IC: $1 \leq |x| < \infty$

- $\operatorname{Cot}^{-1} x$

$$= \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots$$

$$= \frac{\pi}{2} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

IC: $-1 \le x \le 1$

- $\operatorname{Cot}^{-1} x$

$$= \frac{\pi}{2}(1-\operatorname{sgn} x)+\frac{1}{x}-\frac{1}{3x^{3}}+\frac{1}{5x^{5}}-\frac{1}{7x^{7}}+\cdots$$

$$= \frac{\pi}{2}(1-\operatorname{sgn} x)+\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{(2n+1)x^{2n+1}}$$

$\mathrm{IC}:\ 1\leq|x|<\infty$

- $\mathrm{Sec}^{-1}\,x$

$$= \frac{\pi}{2} - \frac{1}{x} - \frac{1}{6}\frac{1}{x^3} - \frac{3}{40}\frac{1}{x^5} - \frac{5}{112}\frac{1}{x^7} - \cdots$$

$$= \frac{\pi}{2} - \frac{1}{x} - \frac{1}{2\cdot 3}\frac{1}{x^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5}\frac{1}{x^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{1}{x^7} - \cdots$$

$$= \frac{\pi}{2} - \frac{1}{x} - \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot \ \cdots \cdot (2n-1)}{2\cdot 4\cdot \ \cdots \ \cdot (2n)(2n+1)}\,\frac{1}{x^{2n+1}}$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}\,\frac{1}{x^{2n+1}}$$

$\mathrm{IC}:\ 1 \le |x| < \infty$

- $\mathrm{Csc}^{-1}\,x$

$$= \frac{1}{x} + \frac{1}{6}\frac{1}{x^3} + \frac{3}{40}\frac{1}{x^5} + \frac{5}{112}\frac{1}{x^7} + \cdots$$

$$= \frac{1}{x} + \frac{1}{2\cdot 3}\frac{1}{x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}\frac{1}{x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{1}{x^7} + \cdots$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot\ \cdots\cdot(2n-1)}{2\cdot 4\cdot\ \cdots\ \cdot(2n)(2n+1)}\ \frac{1}{x^{2n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}\frac{1}{x^{2n+1}}$$

$\mathrm{IC}:\ 1 \le x < \infty$

- $\sinh x$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

IC: $-\infty < x < \infty$

- $\cosh x$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

IC: $-\infty < x < \infty$

- $\tanh x$

$$= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n} - 1\right)B_{2n}}{(2n)!} x^{2n-1}$$

IC: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

- $\coth x$

$$= \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \frac{1}{4725}x^7 + \cdots$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1}$$

IC: $0 < |x| < \pi$

- $\operatorname{sech} x$

$$= 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{61}{720}x^6 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n}$$

$$\text{IC}: \ -\frac{\pi}{2} < x < \frac{\pi}{2}$$

- $\operatorname{csch} x$

$$= \frac{1}{x} - \frac{1}{6}x + \frac{7}{360}x^3 - \frac{31}{15120}x^5 + \frac{127}{604800}x^7 - \cdots$$

$$= \frac{1}{x} - \sum_{n=1}^{\infty} \frac{\left(2^{2n} - 2\right)B_{2n}}{(2n)!} x^{2n-1}$$

$\text{IC}: 0 < |x| < \pi$

- $\sinh^{-1} x$

$$= x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \cdots$$

$$= x - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots$$

$$= x + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 3\cdot \;\cdots\cdot (2n-1)}{2\cdot 4\cdot \;\cdots\;\cdot (2n)(2n+1)} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}$$

IC: $-1 \le x \le 1$

- $\cosh^{-1} x$

$$= \log 2x - \frac{1}{4}\frac{1}{x^2} - \frac{3}{32}\frac{1}{x^4} - \frac{5}{96}\frac{1}{x^6} - \cdots$$

$$= \log 2x - \frac{1}{2\cdot 2}\frac{1}{x^2} - \frac{1\cdot 3}{2\cdot 4\cdot 4}\frac{1}{x^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\frac{1}{x^6} - \cdots$$

$$= \log 2x - \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot \cdots (2n-1)}{2\cdot 4\cdot \cdots \cdot (2n)(2n)}\frac{1}{x^{2n}}$$

$$= \log 2x - \sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2n)}\frac{1}{x^{2n}}$$

IC: $1 \le x < \infty$

- $\tanh^{-1} x$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

IC: $-1 < x < 1$

- $\coth^{-1} x$

$$= \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)x^{2n+1}}$$

IC: $1 < |x| < \infty$

- $\mathrm{Sech}^{-1} x$

$$= \log\frac{2}{x} - \frac{1}{4}x^2 - \frac{3}{32}x^4 - \frac{5}{96}x^6 - \cdots$$

$$= \log\frac{2}{x} - \frac{1}{2\cdot 2}x^2 - \frac{1\cdot 3}{2\cdot 4\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}x^6 - \cdots$$

$$= \log\frac{2}{x} - \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot \;\cdots (2n-1)}{2\cdot 4\cdot \;\cdots\; \cdot (2n)(2n)}\, x^{2n}$$

$$= \log\frac{2}{x} - \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n)}\, x^{2n}$$

IC: $0 < x \le 1$

- $\operatorname{csch}^{-1} x$

$$= \log\frac{2}{x} + \frac{1}{4}x^2 - \frac{3}{32}x^4 + \frac{5}{96}x^6 - \cdots$$

$$= \log\frac{2}{x} + \frac{1}{2\cdot 2}x^2 - \frac{1\cdot 3}{2\cdot 4\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}x^6 - \cdots$$

$$= \log\frac{2}{x} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot \ \cdots (2n-1)}{2\cdot 4\cdot \ \cdots \ \cdot (2n)(2n)}x^{2n}$$

$$= \log\frac{2}{x} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n)!}{2^{2n}(n!)^2(2n)}\ x^{2n}$$

IC: $0 < x \le 1$

- $\operatorname{csch}^{-1} x$

$$= \frac{1}{x} - \frac{1}{6}\frac{1}{x^3} + \frac{3}{40}\frac{1}{x^5} - \frac{5}{112}\frac{1}{x^7} + \cdots$$

$$= \frac{1}{x} - \frac{1}{2\cdot 3}\frac{1}{x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}\frac{1}{x^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{1}{x^7} + \cdots$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 3\cdot \ \cdots \cdot(2n-1)}{2\cdot 4\cdot \ \cdots \ \cdot(2n)(2n+1)} \frac{1}{x^{2n+1}}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} \frac{1}{x^{2n+1}}$$

IC: $1 \le |x| < \infty$