In Pursuit of Perfection:
The Five Platonic Polyhedra

#85 of Gottschalk's Gestalts

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of the Organization & Exposition
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GG85-2

□ here are
a collection of
basic uniform notation & terminology for
and
a collection of
basic uniform facts & formulas about
the five regular polyhedra
= the five regular solids
= the five Platonic polyhedra
= the five Platonic solids

☐ we confine attention to convex polyhedra
and indeed to regular convex polyhedra;
nevertheless much of the notation
and many of the considerations
apply to more general polyhedra;
in the following
the word
' polyhedron'
is taken to mean
' convex polyhedron'

☐ the five
regular / Platonic
polyhedra / solids
are called

- the tetrahedron
- the hexahedron = the cube
- the octahedron
- the dodecahedron
- the icosahedron

when one of the adjective qualifiers
regular or Platonic
is understood;
these designations refer to shape
ie
equivalence = isomorphism
under a similarity transformation;
size ie dimension
must be understood separately
eg specifying the length of an edge

☐ the regular polyhedra
are
the polyhedra with the greatest possible symmetry;
the regular polyhedra
are
the most symmetric polyhedra

☐ the names used in referring to polyhedra
are from Greek words
where
tetra = 4
hexa = 6
octa = 8
dodeca = 12
icosa = 20
poly = many
hedron = face
hedra = faces

☐ general considerations

• considerations take place in
3 - dimensional euclidean space
$=_{ab}$ 3 – space

• a convex polyhedron
$=_{ab}$ a polyhedron
$=_{df}$ a bounded finite intersection
of closed halfspaces of 3 - space
that has nonempty interior

• a polyhedron has
a finite number of
vertices wa points,
edges wa closed line segments,
faces wa convex plane polygons,
facial angles,
dihedral angles,
corner solid angles

- for any polyhedron:

any two intersecting edges
intersect in a vertex;
any two intersecting faces
intersect in an edge;
each edge is bounded by two vertices;
each edge is the intersection of two faces
and is their common side
and is the edge of a dihedral angle
whose sides are these faces;
each face, being a convex polygon,
is bounded by
the union of a finite number of edges
which are its sides;
each face, being a convex polygon,
has a a facial angle at each vertex
whose two sides are sides of the polygon;
the polyhedron has a corner solid angle at each vertex,
the faces with a common vertex being the sides
of the corner solid angle at this vertex

- a polyhedron so far considered

is thought of as
a solid in 3 - space;
but
the surface = the boundary
&
the solid = the boundary plus the interior
determine each other uniquely;
consequently
it is a matter of convenience as to whether
a polyhedron is thought of as
a solid or a surface;
as a solid a polyhedron is topologically a ball;
as a surface a polyhedron is topologically a sphere

- a regular polyhedron

$=_{df}$ a polyhedron
whose self - congruence group
is transitive on
vertices & edges & faces
& facial angles & dihedral angles & corner solid angles;
ie
there is a congruence carrying the polyhedron onto itself
that carries each given
vertex, edge, face,
facial angle, dihedral angle, corner solid angle
onto any other given
vertex, edge, face,
facial angle, dihedral angle, corner solid angle;
this definition contains many redundancies

- it is provable that there are

exactly five regular polyhedra
insofar as shape is concerned;
in a particular category of shape
only size is different eg measured by the edge length

☐ uniform notation & terminology
for any regular polyhedron P

• v

$=_{df}$ the number of the vertices of P

$=_{cl}$ the vertex number of P

wh v ← <u>v</u>ertex

• e

$=_{df}$ the number of the edges of P

$=_{cl}$ the edge number of P

wh e ← <u>e</u>dge

• f

$=_{df}$ the number of the faces of P

$=_{cl}$ the face number of P

wh f ← <u>f</u>ace

- p

$=_{df}$ the number of sides of any face of P
= the number of angles of any face of P
= the number of vertices of any face of P
$=_{cl}$ the facial side number of P

- q

$=_{df}$ the number of faces of P meeting at any vertex of P
= the number of edges of P with common endpoint
at any given vertex of P
$=_{cl}$ the vertex facial number of P

- {p, q}

$=_{df}$ the class of all regular convex polyhedra that have
p as facial side number & q as vertex facial number
= the class of all polyhedra that are similar to P
= the class of all polyhedra with the same shape as P
$=_{cl}$ the similarity class of P

- the notation {p, q}

$=_{cl}$ the Schläfi symbol for P
&
P $\in$ {p, q}

- a

$=_{df}$ the length of any edge of P

$=_{cl}$ the edge length of P

$=_{cl}$ the edge of P

- R

$=_{df}$ the distance from the center of P to any vertex of P

$=$ the radius of the circumsphere of P

$=_{cl}$ the circumradius of P

wh

R ← r̲adius

- ρ

$=_{df}$ the perpendicular distance
from the center of P to any edge of P

$=$ the distance from the center of P
to the midpoint of any edge of P

$=$ the radius of the midsphere of P

$=_{cl}$ the midradius of P

wh

$\rho \leftarrow r \leftarrow$ r̲adius

• r

$=_{df}$ the perpendicular distance

from the center of P to any face of P

= the distance from the center of P

to the center of any face of P

= the radius of the insphere of P

$=_{cl}$ the inradius of P

wh

r ← r̲adius

- R_f

$=_{df}$ the radius of the circumscribed circle of any face of P

$=_{cl}$ the facial circumradius of P

wh

R ← radius & f ← face

- r_f

$=_{df}$ the radius of the inscribed circle of any face of P

$=_{cl}$ the facial inradius of P

wh

r ← radius & f ← face

- A_f

$=_{df}$ the area of any face of P

$=_{cl}$ the single facial area of P

wh

A ← area & f ← face

- A = A_s

$=_{df}$ the total surface area of P

$=_{cl}$ the surface area of P

wh

A ← area & s ← surface

- V

$=_{df}$ the volume of P

wh

V ← volume

- ϑ

$=_{df}$ the measure of the dihedral angle at any edge of P
whose sides are the faces of P with this common edge

$=_{cl}$ the dihedral angle of P

- λ

$=_{df}$ the measure of the angle
subtended by any edge of P from the center of P

$=_{cl}$ the edge angle of P

- ω

$=_{df}$ the measure of the solid angle at any vertex of P
whose boundary is the union of the faces of P
surrounding this vertex
ie that have this point as vertex

$=_{cl}$ the corner solid angle of P

- δ

$=_{df}$ a round angle minus the sum of
the single vertex angles of the faces of P
that surround any given vertex A of P
$=_{cl}$ the angular discrepancy of P at A

- Δ

$=_{df}$ the sum of the angular discrepancies of P
at all vertices of P
$=_{cl}$ the total angular discrepancy of P

- O

$=_{df}$ the unique point that is equidistant from all the vertices of P

= the unique point that is equidistant from all the edges of P

= the unique point that is equidistant from all the faces of P

= the center of the circumsphere of P

= the center of the midsphere of P

= the center of the insphere of P

$=_{cl}$ the circumcenter of P

$=_{cl}$ the midcenter of P

$=_{cl}$ the incenter of P

$=_{cl}$ the center of P

• the circumscribed sphere of P

$=_{cl}$ the circumsphere of P

$=_{df}$ the sphere that passes thru all vertices of P

$=_{dn}$ S(O, R)

• the middle sphere of P

$=_{cl}$ the midsphere of P

$=_{df}$ the sphere that is tangent to all edges of P

= the sphere that passes thru the midpoints of all edges of P

$=_{dn}$ S(O, ρ)

• the inscribed sphere of P

$=_{cl}$ the insphere of P

$=_{df}$ the sphere that is tangent to all faces of P

= the sphere that passes thru the centers of all faces of P

$=_{dn}$ S(O, r)

• wh S(center, radius) denotes

the sphere with the given center and the given radius

& S ← sphere

- the vertex figure of P

$=_{df}$ the regular plane polygon
of q sides / vertices / angles
whose q vertices are
the midpoints of the q edges of P
with common endpoint at any given vertex of P

- a_v

$=_{df}$ the length of a side of the vertex figure of P
$=_{cl}$ the vertex figure side of P

- R_v

$=_{df}$ the circumradius of the vertex figure of P
$=_{cl}$ the vertex figure circumradius of P

- r_v

$=_{df}$ the inradius of the vertex figure of P
$=_{cl}$ the vertex figure inradius of P

- an opposite pair of vertices of P

= a pair of opposite vertices of P

= opposite vertices of P

$=_{df}$ two vertices of P

that are the endpoints of

a circumdiameter of P

ie a diameter of the circumcircle of P

- an opposite pair of edges of P

= a pair of opposite edges of P

= opposite edges of P

$=_{df}$ two edges of P

that are tangent to the midsphere of P

at the midpoints of the edges

which are also the endpoints of

a middiameter of P

ie a diameter of the midsphere of P

- an opposite pair of faces of P

= a pair of opposite faces of P

= opposite faces of P

$=_{df}$ two faces of P

that are tangent to the insphere of P

at the centers of the faces

which are also the endpoints of

an indiameter of P

ie a diameter of the insphere of P;

opposite faces of P

are necessarily parallel

- the dual polyhedron of P

$=_{df}$ the regular polyhedron Q_1
whose vertices are
the centers of the faces of P
&
thus the circumsphere of Q_1
coincides with the insphere of P

= the regular polyhedron Q_2
whose faces are tangent planes
to the circumsphere of P
at the vertices of P
&
thus the insphere of Q_2
coincides with the circumsphere of P

= the regular polyhedron Q_3

whose edges are

the perpendicular - bisector lines of the edges of P

that are tangent to the midsphere of P

&

thus Q_3 and P have a common midsphere;

in this combination of dual regular polyhedra

the vertices of one and the faces of the other

are

in the relation of pole - to - polar

wrt

their common midsphere

note: in passing to the dual

the vertex number & the face number interchange

but the edge number stays constant

☐ four notable right triangles are
associated with any regular polyhedron

• direct attention to:
any given regular polyhedron,
any given face of the polyhedron,
any given edge of the face,
a vertex of the edge;
think of the polyhedron
as resting on the ground
with the face as horizontal base,
with the edge in front,
with the vertex to the left

- mark the vertex as A

- mark the midpoint of the edge as M

- look at the center O of the polyhedron

- draw the segment OO' to the center O' of the face,
OO' being perpendicular to the face

- draw the segment OA
which is above the face except for point A

- the segment OM is perpendicular to the edge
& above the face except for point M

- the segment O' M is perpendicular to the edge
& in the face

- the four notable right triangles

are now to be distinguished as follows:

triangle OO' A with right angle at O'
& perpendicular to the face
& intersecting the face along the side O' A

triangle OO' M with right angle at O'
& perpendicular to the face and to the edge
& intersecting the face along the side O' M

triangle OMA with right angle at M
& lying above the face
except for side MA in the edge

triangle O' MA with right angle at M
& lying in the face with side MA in the edge

- these four right triangles

are the faces
of a special kind of tetrahedron
OO' MA
with lots of right angles
that may be called
the indicative tetrahedron
of the original polyhedron

□ uniform facts & formulas

for any regular polyhedron P

- $v - e + f = 2$ (Euler's formula for polyhedra)

- $qv = 2e = pf$

- $\frac{1}{p} + \frac{1}{q} = \frac{1}{e} + \frac{1}{2}$

- $v = \dfrac{4p}{2p + 2q - pq}$

- $e = \dfrac{2pq}{2p + 2q - pq}$

- $f = \dfrac{4q}{2p + 2q - pq}$

- a little algebraic fact

$0 < 2p + 2q - pq \ = \ 4 - (p-2)(q-2)$

shows

$(p-2)(q-2) < 4$

which restricts $(p,\ q)$

to the five cases

$(3,\ 3),\ (3,\ 4),\ (3,\ 5),\ (4,\ 3),\ (5,\ 3)$

since p & q are integers ≥ 3

- the dual polyhedron of P

$= \ \{q,\ p\}$

- the central side - spanned angle of a face of P

$$= \frac{2\pi}{p}$$

- a vertex angle of a face of P

$$= \pi - \frac{2\pi}{p} = \frac{p-2}{p}\pi$$

- the central side - spanned angle of the vertex figure of P

$$= \frac{2\pi}{q}$$

- a vertex angle of the vertex figure of P

$$= \pi - \frac{2\pi}{q} = \frac{q-2}{q}\pi$$

- $R_f = \frac{a}{2} \csc \frac{\pi}{p}$

- $r_f = \frac{a}{2} \cot \frac{\pi}{p}$

- $a_v = a \cos \frac{\pi}{p}$

- $R_v = \frac{a}{2} \cos \frac{\pi}{p} \csc \frac{\pi}{q}$

- $r_v = \frac{a}{2} \cos \frac{\pi}{p} \cot \frac{\pi}{q}$

- $d^2 =_{df} \sin^2\frac{\pi}{p} - \cos^2\frac{\pi}{q} = \sin^2\frac{\pi}{q} - \cos^2\frac{\pi}{p}$

wh $d > 0$

note that d has the same value for P and its dual

- $R = \frac{a}{2d}\sin\frac{\pi}{q}$

- $\rho = \frac{a}{2d}\cos\frac{\pi}{p}$

- $r = \frac{a}{2d}\cot\frac{\pi}{p}\cos\frac{\pi}{q}$

- $R : r = \tan\frac{\pi}{p}\tan\frac{\pi}{q}$

- $A_f = \frac{1}{4} p a^2 \cot \frac{\pi}{p}$

- $A_s = f A_f$

- $A_s = \frac{1}{4} p f a^2 \cot \frac{\pi}{p}$

- $V = \frac{1}{3} r f A_f$

- $V = \frac{1}{3} r A_s$

- $V = \frac{1}{24d} p f a^3 \cot^2 \frac{\pi}{p} \cos \frac{\pi}{q}$

- $\sin\frac{\vartheta}{2} = \csc\frac{\pi}{p}\cos\frac{\pi}{q}$

- $\cos\vartheta = 1 - 2\csc^2\frac{\pi}{p}\cos^2\frac{\pi}{q}$

- $\cos\frac{\lambda}{2} = \cos\frac{\pi}{p}\csc\frac{\pi}{q}$

- $\cos\lambda = 2\cos^2\frac{\pi}{p}\csc^2\frac{\pi}{q} - 1$

- $\omega = q\vartheta - \pi$

- $\omega = q\cos^{-1}\left(1 - 2\csc^2\frac{\pi}{p}\cos^2\frac{\pi}{q}\right) - \pi$

- $\delta = \dfrac{4\pi^{r}}{v} = \dfrac{720^{o}}{v}$

- $\Delta = 4\pi^{r} = 720^{o}$

- the number of pairs of opposite vertices of $P \neq P_1$

$$= \frac{v}{2}$$

- the number of pairs of opposite edges of P

$$= \frac{e}{2}$$

- the number of pairs of opposite faces of $P \neq P_1$

$$= \frac{f}{2}$$

- the distance between opposite vertices of P

$= 2R$

- the distance between opposite edges of P

$= 2\rho$

- the distance between opposite faces of P

$= 2r$

□ the Schläfli symbol $\{p, q\}$
identifies the regular polyhedron P
viz

- P = tetrahedron $\Leftrightarrow \{p, q\} = \{3, 3\} \Leftrightarrow p = 3 \;\&\; q = 3$
- P = cube $\Leftrightarrow \{p, q\} = \{4, 3\} \Leftrightarrow p = 4 \;\&\; q = 3$
- P = octahedron $\Leftrightarrow \{p, q\} = \{3, 4\} \Leftrightarrow p = 3 \;\&\; q = 4$
- P = dodecahedron $\Leftrightarrow \{p, q\} = \{5, 3\} \Leftrightarrow p = 5 \;\&\; q = 3$
- P = icosahedron $\Leftrightarrow \{p, q\} = \{3, 5\} \Leftrightarrow p = 3 \;\&\; q = 5$

☐ the duals of regular polyhedra

- the dual of the regular polyhedron $\{p, q\}$
is the regular polyhedron $\{q, p\}$

- the dual of the tetrahedron $\{3, 3\}$
is the tetrahedron $\{3, 3\}$;
ie the tetrahedron $\{3, 3\}$ is self - dual

- the dual of the cube $\{4, 3\}$
is the octahedron $\{3, 4\}$

- the dual of the octahedron $\{3, 4\}$
is the cube $\{4, 3\}$

- the dual of the dodecahedron $\{5, 3\}$
is the icosahedron $\{3, 5\}$

- the dual of the icosahedron $\{3, 5\}$
is the dodecahedron $\{5, 3\}$

☐ the tetrahedron P_1

- the regular tetrahedron

$=_{ab}$ the tetrahedron

$=_{dn}$ P_1 wh P ← <u>P</u>latonic polyhedron / solid

$=_{df}$ the regular polyhedron

that has

4 faces

which are congruent equilateral triangles

and st

each vertex of the polyhedron

is the common vertex of 3 faces

and

is the common endpoint of 3 edges

- vertex figure of the tetrahedron

= equilateral triangle

- dual polyhedron of the tetrahedron

= the tetrahedron

ie

the tetrahedron is self - dual

- vertex number = v = 4
- edge number = e = 6
- face number = f = 4

- facial side number = p = 3
- vertex facial number = q = 3
- Schläfi symbol = $\{p, q\}$ = $\{3, 3\}$

- facial circumradius $= R_f = \dfrac{\sqrt{3}}{3}a$

- $R_f : a = \dfrac{\sqrt{3}}{3} = 0.57735+$

- facial inradius $= r_f = \dfrac{\sqrt{3}}{6}a$

- $r_f : a = \dfrac{\sqrt{3}}{6} = 0.28868-$

- single facial area $= A_f = \dfrac{\sqrt{3}}{4}a^2$

- $A_f : a^2 = \dfrac{\sqrt{3}}{4} = 0.43301+$

- vertex figure side $= a_v = \dfrac{1}{2}a$

$a_v : a = \dfrac{1}{2} = 0.5$

- vertex figure circumradius $= R_v = \dfrac{\sqrt{3}}{6}a$

$R_v : a = \dfrac{\sqrt{3}}{6} = 0.28868-$

- vertex figure inradius $= r_v = \dfrac{\sqrt{3}}{12}a$

$r_v : a = \dfrac{\sqrt{3}}{12} = 0.14434-$

- circumradius $= R = \dfrac{\sqrt{6}}{4}a$

- $R : a = \dfrac{\sqrt{6}}{4} = 0.61237+$

- midradius $= \rho = \dfrac{\sqrt{2}}{4}a$

- $\rho : a = \dfrac{\sqrt{2}}{4} = 0.35355+$

- inradius $= r = \dfrac{\sqrt{6}}{12}a$

- $r : a = \dfrac{\sqrt{6}}{12} = 0.20412+$

- $R : \rho : r = 3 : \sqrt{3} : 1$
- $R = \sqrt{3}\rho$
- $R = 3r$
- $\rho = \frac{\sqrt{3}}{3}R$
- $\rho = \sqrt{3}r$
- $r = \frac{1}{3}R$
- $r = \frac{\sqrt{3}}{3}\rho$

- total surface area $= A_s = \sqrt{3}\,a^2$

- $A_s : a^2 = \sqrt{3} = 1.73205+$

- volume $= V = \dfrac{\sqrt{2}}{12}a^3$

- $V : a^3 = \dfrac{\sqrt{2}}{12} = 0.11785+$

- ratio of volume to surface $= \dfrac{V}{A_s} = \dfrac{r}{3} = \dfrac{\sqrt{6}}{36}a$

wh $\dfrac{\sqrt{6}}{36} = 0.06804+$

- $d = \dfrac{\sqrt{2}}{2} = 0.70711-$

- for dihedral angle ϑ

$$\sin\vartheta = \frac{2\sqrt{2}}{3}$$

$$\cos\vartheta = \frac{1}{3}$$

$$\tan\vartheta = 2\sqrt{2}$$

ϑ

$\approx 1.23096^{r}$

$\approx 70.52878^{o}$

$\approx 70^{o}\ 31'\ 43.6''$

- for edge angle λ

$$\sin\lambda \;=\; \frac{2\sqrt{2}}{3}$$

$$\cos\lambda \;=\; -\frac{1}{3}$$

$$\tan\lambda \;=\; -2\sqrt{2}$$

λ

$\approx\;1.91063^{r}$

$\approx\;109.47122^{o}$

$\approx\;109^{o}\;29'\;\;16.4''$

- corner solid angle $= \omega$

$= 3\cos^{-1}\frac{1}{3} - \pi$

$\approx 0.55129^{sr}$

$\approx$ 4.39% of one spacial solid angle $4\pi^{sr} \approx 12.56637^{sr}$

- angular discrepancy $= \delta$

$= \pi^{r} = 180^{o}$

- total angular discrepancy $= \Delta$

$= 4\pi^{r} = 720^{o}$

- the tetrahedron has:

no opposite vertices,
any two vertices being adjacent
in the sense that
they are the endpoints of an edge
&
3 pairs of opposite edges,
each pair being perpendicular
with distance apart $= \frac{\sqrt{2}}{2}a$
wh $\frac{\sqrt{2}}{2} = 0.70710+$
&
no opposite faces,
any two faces being adjacent
in the sense that
their intersection is an edge

• an altitude of the tetrahedron

$=_{df}$ a line segment

from any vertex to the opposite face

(ie the face that does not contain this vertex)

that is perpendicular to the opposite face

• the 4 congruent altitudes of the tetrahedron

meet in the center of the tetrahedron

&

match the 4 vertices and the 4 faces

in pairing opposites;

altitude length

= distance between vertex and opposite face

$= R + r$

$$= \frac{\sqrt{6}}{3} a$$

wh $\frac{\sqrt{6}}{3} = 0.81650 -$

- in the $(x,\ y,\ z)$ rectangular coordinate system

the 4 points

$(1,\ 1,\ 1),\ \ (1, -1, -1),\ \ (-1,\ 1, -1),\ \ (-1, -1,\ 1)$

are the vertices of a regular tetrahedron st

the edge is $2\sqrt{2}$,

the origin is the center,

the edges are face diagonals

of the cube with vertices $(\pm 1, \pm 1, \pm 1)$,

its reflection in the origin

is another congruent regular tetrahedron

with vertices

$(-1, -1, -1),\ \ (-1,\ 1,\ 1),\ \ (1, -1,\ 1),\ \ (1,\ 1,\ -1)$

and whose edges are the other face diagonals

of the cube with vertices $(\pm 1, \pm 1, \pm 1)$;

to get the vertices of the first tetrahedron

start with the point $(1,\ 1,\ 1)$

and reflect in the three coordinate axes;

to get the vertices of the second tetrahedron

start with the reflected - in - the - origin point $(-1, -1, -1)$

and reflect in the three coordinate axes

□ the cube P_2

• the cube
$=_{dn}$ P_2 wh P ← Platonic polyhedron / solid
$=_{df}$ the regular polyhedron
that has
6 faces
which are congruent squares
and st
each vertex of the polyhedron
is the common vertex of 3 faces
and
is the common endpoint of 3 edges

- vertex figure of the cube

= equilateral triangle

- dual polyhedron of the cube

= the octahedron

- vertex number $= v = 8$
- edge number $= e = 12$
- face number $= f = 6$

- facial side number $= p = 4$
- vertex facial number $= q = 3$
- Schläfi symbol $= \{p, q\} = \{4, 3\}$

- facial circumradius $= R_f = \dfrac{\sqrt{2}}{2}a$

- $R_f : a = \dfrac{\sqrt{2}}{2} = 0.70711-$

- facial inradius $= r_f = \dfrac{1}{2}a$

- $r_f : a = \dfrac{1}{2} = 0.5$

- single facial area $= A_f = a^2$

- $A_f : a^2 = 1$

- vertex figure side $= a_v = \dfrac{\sqrt{2}}{2}a$

$$a_v : a = \frac{\sqrt{2}}{2} = 0.70711-$$

- vertex figure circumradius $= R_v = \dfrac{\sqrt{6}}{6}a$

$$R_v : a = \frac{\sqrt{6}}{6} = 0.40825-$$

- vertex figure inradius $= r_v = \dfrac{\sqrt{6}}{12}a$

$$r_v : a = \frac{\sqrt{6}}{12} = 0.20412+$$

- circumradius $= R = \dfrac{\sqrt{3}}{2}a$

- $R : a = \dfrac{\sqrt{3}}{2} = 0.86603-$

- midradius $= \rho = \dfrac{\sqrt{2}}{2}a$

- $\rho : a = \dfrac{\sqrt{2}}{2} = 0.70711-$

- inradius $= r = \dfrac{1}{2}a$

- $r : a = \dfrac{1}{2} = 0.5$

- $R : \rho : r = \sqrt{3} : \sqrt{2} : 1$

- $R = \dfrac{\sqrt{6}}{2}\rho$

- $R = \sqrt{3}\, r$

- $\rho = \dfrac{\sqrt{6}}{3} R$

- $\rho = \sqrt{2}\, r$

- $r = \dfrac{\sqrt{3}}{3} R$

- $r = \dfrac{\sqrt{2}}{2}\rho$

GG85-65 cube

- total surface area $= A_s = 6a^2$

- $A_s : a^2 = 6$

- volume $= V = a^3$

- $V : a^3 = 1$

- ratio of volume to surface $= \dfrac{V}{A_s} = \dfrac{r}{3} = \dfrac{1}{6}a$

wh $\dfrac{1}{6} = 0.16667-$

- $d = \dfrac{1}{2} = 0.5$

GG85-66 cube

- for dihedral angle ϑ

$\sin\vartheta = 1$

$\cos\vartheta = 0$

ϑ

$= \dfrac{\pi^{r}}{2}$

$\approx 1.57080^{r}$

$= 90^{o}$

- for edge angle λ

$$\sin\lambda = \frac{2\sqrt{2}}{3}$$

$$\cos\lambda = \frac{1}{3}$$

$$\tan\lambda = 2\sqrt{2}$$

λ

$\approx 1.23096^{r}$

$\approx 70.52878^{o}$

$\approx 70^{o}\ 31'\ 43.6''$

- corner solid angle $= \omega$

$= \dfrac{\pi^{sr}}{2}$

$=$ one octant

$=$ one - eighth of one spacial solid angle $4\pi^{sr} \approx 12.56637^{sr}$

$\approx 1.57080^{sr}$

- angular discrepancy $= \delta$

$= \dfrac{\pi^{r}}{2} = 90^{o}$

- total angular discrepancy $= \Delta$

$= 4\pi^{r} = 720^{o}$

- the cube has:

4 pairs of opposite vertices,

each pair with distance apart $= \sqrt{3}\,a$

wh $\sqrt{3} = 1.73205+$

&

6 pairs of opposite sides,

each pair being parallel

with distance apart $= \sqrt{2}\,a$

wh $\sqrt{2} = 1.41421+$

&

3 pairs of opposite faces,

each pair being parallel

with distance apart $= a$

- in the $(x,\ y,\ z)$ rectangular coordinate system

the 8 points

$(\pm 1,\ \pm 1,\ \pm 1)$

are the vertices of a cube st

the edge is 2,

the origin is the center,

the coordinate axes are axes of symmetry,

the coordinate planes are planes of symmetry,

the vertices are ' symmetrically located'

in the 8 octants,

the edges are parallel to the coordinate axes,

the faces are parallel to the coordinate planes,

the surface is the graph of the equation

$\max\{|x|,\ |y|,\ |z|\} = 1$,

the interior is the graph of the strict inequality

$\max\{|x|,\ |y|,\ |z|\} < 1$,

the entirety is the graph of the weak inequality

$\max\{|x|,\ |y|,\ |z|\} \le 1$;

to get the vertices of this cube
start with the point (1, 1, 1)
and reflect in
the origin,
each of the three coordinate axes,
each of the three coordinate planes

☐ the octahedron P_3

- the regular octahedron

$=_{ab}$ the octahedron

$=_{dn}$ P_3 wh P ← Platonic polyhedron / solid

$=_{df}$ the regular polyhedron

that has

8 faces

which are congruent equilateral triangles

and st

each vertex of the polyhedron

is the common vertex of 4 faces

and

is the common endpoint of 4 edges

- vertex figure of the octahedron

= square

- dual polyhedron of the octahedron

= the cube

- vertex number $= v = 6$
- edge number $= e = 12$
- face number $= f = 8$

- facial side number $= p = 3$
- vertex facial number $= q = 4$
- Schläfi symbol $= \{p, q\} = \{3, 4\}$

- facial circumradius $= R_f = \dfrac{\sqrt{3}}{3}a$

- $R_f : a = \dfrac{\sqrt{3}}{3} = 0.57735+$

- facial inradius $= r_f = \dfrac{\sqrt{3}}{6}a$

- $r_f : a = \dfrac{\sqrt{3}}{6} = 0.28868-$

- single facial area $= A_f = \dfrac{\sqrt{3}}{4}a^2$

- $A_f : a^2 = \dfrac{\sqrt{3}}{4} = 0.43301+$

- vertex figure side $= a_v = \frac{1}{2}a$

$$a_v : a = \frac{1}{2} = 0.5$$

- vertex figure circumradius $= R_v = \frac{\sqrt{2}}{4}a$

$$R_v : a = \frac{\sqrt{2}}{4} = 0.35355+$$

- vertex figure inradius $= r_v = \frac{1}{4}a$

$$r_v : a = \frac{1}{4} = 0.25$$

- circumradius $= R = \dfrac{\sqrt{2}}{2}a$
- $R : a = \dfrac{\sqrt{2}}{2} = 0.70711-$

- midradius $= \rho = \dfrac{1}{2}a$

- $\rho : a = \dfrac{1}{2} = 0.5$

- inradius $= r = \dfrac{\sqrt{6}}{6}a$

- $r : a = \dfrac{\sqrt{6}}{6} = 0.40825-$

- $R : \rho : r = \sqrt{6} : \sqrt{3} : \sqrt{2}$

- $R = \sqrt{2}\,\rho$

- $R = \sqrt{3}\,r$

- $\rho = \dfrac{\sqrt{2}}{2} R$

- $\rho = \dfrac{\sqrt{6}}{2} r$

- $r = \dfrac{\sqrt{3}}{3} R$

- $r = \dfrac{\sqrt{6}}{3} \rho$

- total surface area $= A_s = 2\sqrt{3}\,a^2$

- $A_s : a^2 = 2\sqrt{3} = 3.46410+$

- volume $= V = \dfrac{\sqrt{2}}{3}a^3$

- $V : a^3 = \dfrac{\sqrt{2}}{3} = 0.47140+$

- ratio of volume to surface $= \dfrac{V}{A_s} = \dfrac{r}{3} = \dfrac{\sqrt{6}}{18}a$

wh $\dfrac{\sqrt{6}}{18} = 0.13608+$

- $d = \dfrac{1}{2} = 0.5$

GG85-80 octahedron

- for dihedral angle ϑ

$$\sin\vartheta = \frac{2\sqrt{2}}{3}$$

$$\cos\vartheta = -\frac{1}{3}$$

$$\tan\vartheta = -2\sqrt{2}$$

ϑ

$\approx 1.91063^{r}$

$\approx 109.47122^{o}$

$\approx 109^{o}\ 29'\ 16.4''$

- for edge angle λ

$\sin\lambda = 1$

$\cos\lambda = 0$

λ

$= \dfrac{\pi^{r}}{2}$

$\approx 1.57080^{r}$

$= 90^{o}$

• corner solid angle $= \omega$

$= 4\cos^{-1}\frac{1}{3} - \pi$

$\approx 1.78216^{sr}$

$\approx$ 14.18% of one spacial solid angle $4\pi^{sr} \approx 12.56637^{sr}$

• angular discrepancy $= \delta$

$= \dfrac{2\pi^{r}}{3} = 120^{o}$

• total angular discrepancy $= \Delta$

$= 4\pi^{r} = 720^{o}$

- the octahedron has:

3 pairs of opposite vertices,

each pair with distance apart $= \sqrt{2}\,a$

wh $\sqrt{2} = 1.41421+$

&

6 pairs of opposite sides,

each pair being parallel

with distance apart $= a$

&

4 pairs of opposite faces,

each pair being parallel

with distance apart $= \dfrac{\sqrt{6}}{3}\,a$

wh $\dfrac{\sqrt{6}}{3} = 0.81650-$

- in the $(x,\ y,\ z)$ rectangular coordinate system

the 6 points

$(\pm 1,\ 0,\ 0),\ \ (0,\ \pm 1,\ 0),\ \ (0,\ 0,\ \pm 1)$

are the vertices of a regular octahedron st

the edge is $\sqrt{2}$,

the origin is the center,

the coordinate axes are axes of symmetry,

the coordinate planes are planes of symmetry,

the vertices are ' symmetrically located'

on the eight coordinate half - axes,

the surface is the graph of the equation

$|x| + |y| + |z| \ = \ 1,$

the interior is the graph of the strict inequality

$|x| + |y| + |z| \ < \ 1,$

the entirety is the graph of the weak inequality

$|x| + |y| + |z| \ \leq \ 1,$

the vertices are the centers of the faces of the cube

whose vertices are $(\pm 1,\ \pm 1,\ \pm 1)$

□ the dodecahedron P_4

• the regular dodecahedron

$=_{ab}$ the dodecahedron

$=_{dn}$ P_4 wh P ← Platonic polyhedron / solid

$=_{df}$ the regular polyhedron

that has

12 faces

which are congruent regular pentagons

and st

each vertex of the polyhedron

is the common vertex of 3 faces

and

is the common endpoint of 3 edges

- vertex figure of the dodecahedron

= equilateral triangle

- dual polyhedron of the dodecahedron

= the icosahedron

- vertex number = v = 20
- edge number = e = 30
- face number = f = 12

- facial side number = p = 5
- vertex facial number = q = 3
- Schläfi symbol = {p, q} = {5, 3}

- facial circumradius $= R_f = \dfrac{\sqrt{50+10\sqrt{5}}}{10}\,a$

- $R_f : a = \dfrac{\sqrt{50+10\sqrt{5}}}{10} = 0.85065+$

- facial inradius $= r_f = \dfrac{\sqrt{25+10\sqrt{5}}}{10}\,a$

- $r_f : a = \dfrac{\sqrt{25+10\sqrt{5}}}{10} = 0.68819+$

- single facial area $= A_f = \dfrac{\sqrt{25+10\sqrt{5}}}{4}\,a^2$

- $A_f : a^2 = \dfrac{\sqrt{25+10\sqrt{5}}}{4} = 1.72048-$

- vertex figure side $= a_v = \dfrac{1+\sqrt{5}}{4}a$

$$a_v : a = \frac{1+\sqrt{5}}{4} = 0.80902-$$

- vertex figure circumradius $= R_v = \dfrac{\sqrt{3}+\sqrt{15}}{12}a$

$$R_v : a = \frac{\sqrt{3}+\sqrt{15}}{12} = 0.46709-$$

- vertex figure inradius $= r_v = \dfrac{\sqrt{3}+\sqrt{15}}{24}a$

$$r_v : a = \frac{\sqrt{3}+\sqrt{15}}{24} = 0.23354+$$

- circumradius $= R = \dfrac{\sqrt{3}+\sqrt{15}}{4}a$

- $R : a = \dfrac{\sqrt{3}+\sqrt{15}}{4} = 1.40126-$

- midradius $= \rho = \dfrac{3+\sqrt{5}}{4}a$

- $\rho : a = \dfrac{3+\sqrt{5}}{4} = 1.30902-$

- inradius $= r = \dfrac{\sqrt{250+110\sqrt{5}}}{20}a$

- $r : a = \dfrac{\sqrt{250+110\sqrt{5}}}{20} = 1.11352-$

- total surface area $= A_s = 3\sqrt{25+10\sqrt{5}}\,a^2$

- $A_s : a^2 = 3\sqrt{25+10\sqrt{5}} = 20.64573-$

- volume $= V = \dfrac{15+7\sqrt{5}}{4}\,a^3$

- $V : a^3 = \dfrac{15+7\sqrt{5}}{4} = 7.66312-$

- ratio of volume to surface $= \dfrac{V}{A_s} = \dfrac{r}{3} = \dfrac{\sqrt{250+110\sqrt{5}}}{60}\,a$

wh $\dfrac{\sqrt{250+110\sqrt{5}}}{60} = 0.37117+$

- $d = \dfrac{\sqrt{5}-1}{4} = 0.30902-$

- for dihedral angle ϑ

$$\sin\vartheta = \frac{2\sqrt{5}}{5}$$

$$\cos\vartheta = -\frac{\sqrt{5}}{5}$$

$$\tan\vartheta = -2$$

ϑ

$\approx 2.03444^{r}$

$\approx 116.56505^{o}$

$\approx 116^{o}\ 33'\ 54.2''$

- for edge angle λ

$$\sin\lambda = \frac{2}{3}$$

$$\cos\lambda = \frac{\sqrt{5}}{3}$$

$$\tan\lambda = \frac{2\sqrt{5}}{5}$$

λ

$\approx 0.72973^{r}$

$\approx 41.81031^{o}$

$\approx 41^{o}\ 48'\ 37.1''$

- corner solid angle $= \omega$

$$= 3\cos^{-1}\left(-\frac{\sqrt{5}}{5}\right) - \pi$$

$$\approx 2.96173^{sr}$$

$\approx$ 23.57% of one spacial solid angle $4\pi^{sr} \approx 12.56637^{sr}$

- angular discrepancy $= \delta$

$$= \frac{\pi^{r}}{5} = 36^{o}$$

- total angular discrepancy $= \Delta$

$$= 4\pi^{r} = 720^{o}$$

- the dodecahedron has:

10 pairs of opposite vertices,

each pair with distance apart $= \dfrac{\sqrt{3}+\sqrt{15}}{2}\,a$

wh $\dfrac{\sqrt{3}+\sqrt{15}}{2} = 2.80252-$

&

15 pairs of opposite sides,

each pair being parallel

with distance apart $= \dfrac{3+\sqrt{5}}{2}\,a$

wh $\dfrac{3+\sqrt{5}}{2} = 2.61803+$

&

6 pairs of opposite faces,

each pair being parallel

with distance apart $= \dfrac{\sqrt{250+110\sqrt{5}}}{10}\,a$

wh $\dfrac{\sqrt{250+110\sqrt{5}}}{10} = 2.22703+$

- let

$$\varphi =_{\text{df}} \frac{1}{2}\left(1+\sqrt{5}\right) =_{\text{cl}} \text{ the golden ratio}$$

then in the $(x,\ y,\ z)$ rectangular coordinate system

the 20 points

$$(\pm 1, \pm 1, \pm 1)$$

$$\left(0,\ \pm\varphi,\ \pm\frac{1}{\varphi}\right)$$

$$\left(\pm\frac{1}{\varphi},\ 0,\ \pm\varphi\right)$$

$$\left(\pm\varphi,\ \pm\frac{1}{\varphi},\ 0\right)$$

are the vertices of a regular dodecahedron

with edge $\sqrt{5}\ -1$

- the following circuit of vertices makes a face of the dodecahedron; the other faces are obtained by change of sign and cyclic change of the columns of coordinates;

$(1,\ 1,\ 1)$

$$\left(0,\ \varphi,\ \frac{1}{\varphi}\right)$$

$$\left(0,\ \varphi, -\frac{1}{\varphi}\right)$$

$(1,\ 1, -1)$

$$\left(\varphi,\ \frac{1}{\varphi},\ 0\right)$$

$(1,\ 1,\ 1)$

- the center of the dodecahedron is the origin;
the edges of the cube with vertices
$(\pm 1, \pm 1, \ \pm 1)$ are face diagonals
of the dodecahedron;
the face diagonals have length 2

- the diagonals of the faces
of a regular dodecahedron
form the edges of five cubes;
each of the five diagonals of a given face
serves as the edge of one cube;
each of the five cubes has an edge
contained in any given face

☐ the icosahedron P_5

• the regular icosahedron
$=_{ab}$ the icosahedron
$=_{dn}$ P_5 wh P ← Platonic polyhedron / solid
$=_{df}$ the regular polyhedron
that has
20 faces
which are congruent equilateral triangles
and st
each vertex of the polyhedron
is the common vertex of 5 faces
and
is the common endpoint of 5 edges

- vertex figure of the icosahedron

= regular pentagon

- dual polyhedron of the icosahedron

= the dodecahedron

- vertex number = v = 12
- edge number = e = 30
- face number = f = 20

- facial side number = p = 3
- vertex facial number = q = 5
- Schläfi symbol = {p, q} = {3, 5}

- facial circumradius $= R_f = \frac{\sqrt{3}}{3}a$

- $R_f : a = \frac{\sqrt{3}}{3} = 0.57735+$

- facial inradius $= r_f = \frac{\sqrt{3}}{6}a$

- $r_f : a = \frac{\sqrt{3}}{6} = 0.28868-$

- single facial area $= A_f = \frac{\sqrt{3}}{4}a^2$

- $A_f : a^2 = \frac{\sqrt{3}}{4} = 0.43301+$

- vertex figure side $= a_v = \frac{1}{2}a$

$$a_v : a = \frac{1}{2} = 0.5$$

- vertex figure circumradius $= R_v = \frac{\sqrt{50+10\sqrt{5}}}{20}a$

$$R_v : a = \frac{\sqrt{50+10\sqrt{5}}}{20} = 0.42533-$$

- vertex figure inradius $= r_v = \frac{\sqrt{25+10\sqrt{5}}}{20}a$

$$r_v : a = \frac{\sqrt{25+10\sqrt{5}}}{20} = 0.34410-$$

- circumradius $= R = \dfrac{\sqrt{10+2\sqrt{5}}}{4}a$

- $R : a = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = 0.95106-$

- midradius $= \rho = \dfrac{1+\sqrt{5}}{4}a$

- $\rho : a = \dfrac{1+\sqrt{5}}{4} = 0.80902-$

- inradius $= r = \dfrac{3\sqrt{3}+\sqrt{15}}{12}a$

- $r : a = \dfrac{3\sqrt{3}+\sqrt{15}}{12} = 0.75576+$

- total surface area $= A_s = 5\sqrt{3}\,a^2$

- $A_s : a^2 = 5\sqrt{3} = 8.66025+$

- volume $= V = \dfrac{5\left(3+\sqrt{5}\right)}{12}a^3$

- $V : a^3 = \dfrac{5\left(3+\sqrt{5}\right)}{12} = 2.18169+$

- ratio of volume to surface $= \dfrac{V}{A_s} = \dfrac{r}{3} = \dfrac{3\sqrt{3}+\sqrt{15}}{36}a$

wh $\dfrac{3\sqrt{3}+\sqrt{15}}{36} = 0.25192+$

- $d = \dfrac{\sqrt{5}-1}{4} = 0.30902-$

GG85-106 icosahedron

- for dihedral angle ϑ

$$\sin\vartheta = \frac{2}{3}$$

$$\cos\vartheta = -\frac{\sqrt{5}}{3}$$

$$\tan\vartheta = -\frac{2\sqrt{5}}{5}$$

ϑ

$\approx 2.41186^{r}$

$\approx 138.18969^{o}$

$\approx 138^{o}\ 11'\ 22.9''$

- for edge angle λ

$$\sin\lambda = \frac{2\sqrt{5}}{5}$$

$$\cos\lambda = \frac{\sqrt{5}}{5}$$

$$\tan\lambda = 2$$

λ

$\approx 1.10715^{r}$

$\approx 63.43495^{o}$

$\approx 63^{o}\ 22'\ 05.8''$

- corner solid angle $= \omega$

$$= 5\cos^{-1}\left(-\frac{\sqrt{5}}{3}\right) - \pi$$

$$\approx 8.91773^{sr}$$

$\approx$ 70.97% of one spacial solid angle $4\pi^{sr} \approx 12.56637^{sr}$

- angular discrepancy $= \delta$

$$= \frac{\pi^r}{3} = 60^o$$

- total angular discrepancy $= \Delta$

$$= 4\pi^r = 720^o$$

• the icosahedron has:

6 pairs of opposite vertices,

each pair with distance apart $= \dfrac{\sqrt{10 + 2\sqrt{5}}}{2}\,a$

wh $\dfrac{\sqrt{10 + 2\sqrt{5}}}{2} = 1.90211+$

&

15 pairs of opposite sides,

each pair being parallel

with distance apart $= \dfrac{1 + \sqrt{5}}{2}\,a$

wh $\dfrac{1 + \sqrt{5}}{2} = \varphi = 1.61803+$

&

10 pairs of opposite faces,

each pair being parallel

with distance apart $= \dfrac{3\sqrt{3} + \sqrt{15}}{6}\,a$

wh $\dfrac{3\sqrt{3} + \sqrt{15}}{6} = 1.51152+$

- let

$\varphi \ =_{\mathrm{df}} \ \frac{1}{2}\left(1+\sqrt{5}\right) \ =_{\mathrm{cl}}$ the golden ratio

then in the $(x,\ y,\ z)$ rectangular coordinate system

the 12 points

$(0,\ \pm 1,\ \pm\varphi)$

$(\pm\varphi,\ 0,\ \pm 1)$

$(\pm 1,\ \pm\varphi,\ 0)$

are the vertices of a regular icosahedron

with edge 2

• the following circuit of vertices
makes a face of the above icosahedron;
the other faces are obtained by
change of sign and cyclic change
of the columns of coordinates;

$(0,\ 1,\ \varphi)$

$(1,\ \varphi,\ 0)$

$(\varphi,\ 0,\ 1)$

$(0,\ 1,\ \varphi)$

- the 12 vertices of a regular icosahedron

can be partitioned into
three subsets of four elements each
so that each subset consists of
the vertices of a golden rectangle
with the three golden rectangles
mutually perpendicular;
in the above example
a subset consists of the four vertices
with the same coordinate equal to 0

- there are two points on any given line segment
that divide the segment in golden ratio
because there are two orientations of the segment;
mark the 24 points of golden ratio division
on the 12 edges of a regular octahedron;
these 24 points are the vertices
of two congruent regular icosahedra

☐ relations between dual regular polyhedra

let the quantity designations

a, R, ρ, r, A, V, ϑ, λ

subscripted by

t, c, o, d, i

refer by initial letter to the regular polyhedra:

tetrahedron,

cube,

octahedron,

dodecahedron,

icosahedron

then

- for two dual regular polyhedra

the product of
the circumradius of one times the inradius of the other
equals
the product of their two midradii;
inp

* for the self – dual tetrahedron

$$R_t r_t = \rho_t^2 = \frac{1}{8} a_t^2$$

wh $\frac{1}{8} = 0.125$

* for the dual cube & octahedron

$$R_c r_o = \rho_c \rho_o = R_o r_c = \frac{\sqrt{2}}{4} a_c a_o$$

wh $\frac{\sqrt{2}}{4} = 0.35355+$

* for the dual dodecahedron & icosahedron

$$R_d r_i = \rho_d \rho_i = R_i r_d = \frac{2+\sqrt{5}}{4} a_d a_i$$

wh $\frac{2+\sqrt{5}}{4} = 1.05902-$

• for two dual regular polyhedra
the ratio of the circumradius to the inradius
has the same value;
inp

* for the self - dual tetrahedron

$$R_t : r_t = 3$$

* for the dual cube & octahedron

$$R_c : r_c = R_o : r_o = \sqrt{3} = 1.73205+$$

• for the dual dodecahedron & icosahedron

$$R_d : r_d = R_i : r_i = \sqrt{15 - 6\sqrt{5}} = 1.25841-$$

• the dual cube & octahedron

have the same

* circumradius ie $R_c = R_o$

$\Leftrightarrow$

$$a_c : a_o = \frac{\sqrt{6}}{3} \Leftrightarrow a_o : a_c = \frac{\sqrt{6}}{2}$$

* midradius ie $\rho_c = \rho_o$

$\Leftrightarrow$

$$a_c : a_o = \frac{\sqrt{2}}{2} \Leftrightarrow a_o : a_c = \sqrt{2}$$

* inradius ie $r_c = r_o$

$\Leftrightarrow$

$$a_c : a_o = \frac{\sqrt{6}}{3} \Leftrightarrow a_o : a_c = \frac{\sqrt{6}}{2}$$

• the dual dodecahedron & icosahedron

have the same

* circumradius ie $R_d = R_i$

$\Leftrightarrow$

$$a_d : a_i = \frac{\sqrt{30 - 6\sqrt{5}}}{6} \Leftrightarrow a_i : a_d = \frac{\sqrt{150 + 30\sqrt{5}}}{10}$$

* midradius ie $\rho_d = \rho_i$

$\Leftrightarrow$

$$a_d : a_i = \frac{\sqrt{5} - 1}{2} = \frac{1}{\varphi} \Leftrightarrow a_i : a_d = \frac{1 + \sqrt{5}}{2} = \varphi$$

* inradius ie $r_d = r_i$

$\Leftrightarrow$

$$a_d : a_i = \frac{\sqrt{30 - 6\sqrt{5}}}{6} \Leftrightarrow a_i : a_d = \frac{\sqrt{150 + 30\sqrt{5}}}{10}$$

- for two dual regular polyhedra

tfsape

* they have the same circumradius

* they have the same inradius

* they have the same ratio of volume to surface area
and in this case the ratio equals one - third the inradius

* the ratio of their volumes
equals
the ratio of their surface areas

• for the dual cube & octahedron

tfsape:

* they have the same circumradius

ie

$R_c = R_o$

* they have the same inradius

ie

$r_c = r_o$

* $a_c : a_o = \dfrac{\sqrt{6}}{3}$

* $a_o : a_c = \dfrac{\sqrt{6}}{2}$

* they have the same ratio of volume to surface area

ie

$V_c : A_c = V_o : A_o$

and in this case the ratio $= \frac{r_c}{3} = \frac{r_o}{3}$

* the ratio of their volumes

equals

the ratio of their surface areas

ie

$V_c : V_o = A_c : A_o$

and in this case the ratio $= \frac{2\sqrt{3}}{3} = 1.15470+$

• for the dual dodecahedron & icosahedron

tfsape:

* they have the same circumradius

ie

$R_d = R_i$

* they have the same inradius

ie

$r_d = r_i$

* $a_d : a_i = \dfrac{\sqrt{30 - 6\sqrt{5}}}{6}$

* $a_i : a_d = \dfrac{\sqrt{150 + 30\sqrt{5}}}{10}$

* they have the same ratio of volume to surface area

ie

$$V_d : A_d = V_i : A_i$$

and in this case the ratio $= \dfrac{r_d}{3} = \dfrac{r_i}{3}$

* the ratio of their volumes

equals

the ratio of their surface areas

ie

$$V_d : V_i = A_d : A_i$$

and in this case the ratio $= \dfrac{\sqrt{30 + 6\sqrt{5}}}{6} = 0.91852+$

- for two dual regular polyhedra

the dihedral angle of one

and

the edge angle of the other

are supplementary;

inp

* for the self - dual tetrahedron

$$\vartheta_t + \lambda_t = \text{str ang}$$

$$\vartheta_t = \cos^{-1}\frac{1}{3} \approx 70.5^{\circ}$$

$$\lambda_t = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109.5^{\circ}$$

* for the dual cube & octahedron

$\vartheta_c + \lambda_o = \text{str ang}$

$\vartheta_o + \lambda_c = \text{str ang}$

$\vartheta_c = \text{rt ang} = 90^o$

$\vartheta_o = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109.5^o$

$\lambda_c = \cos^{-1}\frac{1}{3} \approx 70.5^o$

$\lambda_o = \text{rt ang} = 90^o$

* for the dual dodecahedron & icosahedron

$\vartheta_d + \lambda_i = \text{str ang}$

$\vartheta_i + \lambda_d = \text{str ang}$

$\vartheta_d = \cos^{-1}\left(-\frac{\sqrt{5}}{5}\right) \approx 116.6^o$

$\vartheta_i = \cos^{-1}\left(-\frac{\sqrt{5}}{3}\right) \approx 138.2^o$

$\lambda_d = \cos^{-1}\frac{\sqrt{5}}{3} \approx 41.8^o$

$\lambda_i = \cos^{-1}\frac{\sqrt{5}}{5} \approx 63.4^o$

HN.

☐ the five regular polyhedra
were known to the ancient Greeks;
according to historical records
Pythagoras knew of
the tetrahedron & the cube & the dodecahadron;
Theaetetus was the first to study
the octahedron and the icosahedron

these five solids were described by Plato ca 350 BCE
in his dialogue Timaeus where he equated
four of them with the four ‘primitive elements’
viz
tetrahedron = fire
cube = earth
octahedron = air
icosahedron = water

also Plato took
the dodecahedron to correspond to
the entire cosmos/universe/ether/heavens
which he called
‘the fifth essence’ = Quinta Essentia (Latin)

☐ it was considered by the ancient Greeks
that all matter is made up of
a combination of these four elements;
from a present-day scientific point of view,
this belief appears to be
without any rational justification or interpretation;
however, if we direct attention to
the four states of matter,
the situation changes;
consider the correspondence:
solid = earth
liquid = water
gas = air
plasma = fire

☐ Euclid's The Elements, composed ca 300 BCE,
began with the construction of
an equilateral triangle
and ended with the construction of
the five Platonic polyhedra;
thus it has been remarked, somewhat humorously,
that The Elements may be considered to be
a rather lengthy construction
by ruler and compasses
of the five regular polyhedra
instead of
an introduction to & summary of
elementary geometry

☐ Kepler gave the following explanation
for the association given above
between the regular polyhedra and the primitive elements

• it seems by just looking at them
that of all the regular polyhedra,
the tetrahedron has the least volume
for its given surface area
and
the regular icosahedron has the greatest volume
for its given surface area;
thus the tetrahedron exhibits the quality of dryness
and
the icosahedron exhibits the quality of wetness;
dryness corresponds to Fire
and
wetness corresponds to Water;
note that when an object that contains water
begins to dry out,
it becomes smaller in volume
but retains its surface area (more or less);
think of going
from a grape to a raisin
and
from a plumb to a prune

• the cube stands firmly on its square base
and corresponds to the stable Earth

• the octahedron rotates freely
when held at opposite vertices by the finger tips;
hence the octahedron is
the least stable of the regular polyhedra
and corresponds to the mobile Air

• the docecahedron has twelve faces
and the Zodiac has twelve signs/constellations;
thus the dodecahedron corresponds to the Universe

☐ Kepler used the five regular polyhedra
to construct a geometric model of the solar system
as it was known at his time;
the six planets then known,
listed outwards from the Sun,
are
Mercury
Venus
Earth
Mars
Jupiter
Saturn

think of six spherical shells
with the Sun as common center
and containing the six planetary orbits
in the thickness of the shells;
the spherical shells are separated by
the five regular polyhedra as follows:

the octahedron separates by contact the shells
containing the orbits of Mercury and Venus

the icosahedron separates by contact the shells
containing the orbits of Venus and Earth

the dodecahedron separates by contact the shells
containing the orbits of Earth and Mars

the tetrahedron separates by contact the shells
containing the orbits of Mars and Jupiter

the cube separates by contact the shells
containing the orbits of Jupiter and Saturn

each polyhedron is enclosed between
an outer shell and an inner shell;
each polyhedron has
the inner sphere of the outer shell
as its circumsphere
and
the outer sphere of the inner shell
as its insphere

this geometric configuration
is more poetry than astronomy;
it is not even close to
the ratios of the actual distances,
as Kepler himself began to realize

☐ biolines

• Pythagoras of Samos
ca 580 - ca 500 BCE
Greek
mathematician, philosopher, sage, mystic;
made many basic mathematical discoveries;
founded the Pythagorean School
which taught a way of life;
profoundly influential on Plato
and later philosophers

• Plato of Athens
ca 428 - ca 348 BCE
Greek
philosopher;
one of the most important philosophers of all time;
pupil of Socrates;
teacher of Aristotle

• Theaetetus of Athens
ca 414 - ca 369 BCE
Greek
mathematician;
associate of Plato
who named the dialogue Theaetetus after him

• Euclid of Alexandria
fl 300 BCE
Greek
mathematician;
author of The Elements,
the best known mathematical treatise of all time,
the longest lasting geometry textbook of all time
being used up to the early part of the 20th century,
the first mathematics book to be printed,
the first known axiomatic treatment of mathematics

• Johann Kepler
1571-1630
German
astronomer, mathematician, physicist, mystic;
derived empirically his three laws of planetary motion;
modern astronomy begins with his work

□ all five of the regular polyhedra
occur in nature;
the tetrahedron & the cube & the octahedron
appear in certain crystals;
the dodecahedron & the icosahedron
appear in certain viruses
and in the minute marine protozoans
of the class Radiolaria

□ physical models
of the dodedcahedron & the icosahedron
were made long ago;
a stone dodecahedron dating back to prehistoric times
was found in northern Italy;
the ancient Romans played with icosahedral dice;
basalt and quartz icosahedral dice
from the Ptolemaic Period of Egypt
ca first century BCE are extant;
ancient Celtic bronze models of the dodecahedron
are in various museums

□ the esthetic and intellectual appeal
of the five regular polyhedra
has inspired and fascinated
architects & artists & artisans
& savants & scholars & scientists
from ancient times to the present;
in the pursuit of perfection
the Platonic polyhedra
are a beacon, a goal, and a pointer along the road

□ trig fcns of $\frac{\pi}{3} = 60^{o}$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = 0.86603-$$

$$\cos\frac{\pi}{3} = \frac{1}{2} = 0.5$$

$$\tan\frac{\pi}{3} = \sqrt{3} = 1.73205+$$

$$\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}} = 0.57735+$$

$$\sec\frac{\pi}{3} = 2$$

$$\csc\frac{\pi}{3} = \frac{2}{\sqrt{3}} = 1.15470+$$

□ trig fcns of $\frac{\pi}{4} = 45^o$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.70711-$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.70711-$$

$$\tan\frac{\pi}{4} = 1$$

$$\cot\frac{\pi}{4} = 1$$

$$\sec\frac{\pi}{4} = \sqrt{2} = 1.41421+$$

$$\csc\frac{\pi}{4} = \sqrt{2} = 1.41421+$$

□ trig fcns of $\frac{\pi}{5} = 36^{\text{o}}$

$$\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} = 0.58779-$$

$$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4} = 0.80902-$$

$$\tan\frac{\pi}{5} = \sqrt{5-2\sqrt{5}} = 0.72654+$$

$$\cot\frac{\pi}{5} = \frac{\sqrt{25+10\sqrt{5}}}{5} = 1.37638+$$

$$\sec\frac{\pi}{5} = \sqrt{5}-1 = 1.23607-$$

$$\csc\frac{\pi}{5} = \frac{\sqrt{50+10\sqrt{5}}}{5} = 1.70130+$$

□ nets & plans of polyhedra

• thinking of a regular polyhedron
as having rigid plane polygonal plates as faces
and as having edges that may be cut
or provided with hinges at will,
it is possible to unfold a given regular polyhedron
into a plane polygon with boundary
which may be called
a net of the polyhedron;
conversely
given a net of a regular polyhedron,
the net may be folded together
with matched edges glued on the way
so that the original polyhedron
may be recovered;
the process is more physics than mathematics
(call it kinetic mathematics)
but it is a powerful aid
to the visualiztion of a polyhedron

- a plan of an object = polyhedron / solid

$=_{df}$ a perpendicularly projected image
of the object on a plane

= an overhead view of the object

- the words

plan, plain, plane

have a common Latin origin

viz

planus (adj) = level, flat

planum (noun) = level ground