Triple-Treat Triangles

#84 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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GG84-1 (63)

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GG84-2

□ by a triple - treat triangle
we here mean
a triangle that has
good angles
&
good trig fcns of angles
&
good ratio of sides;
by good
we here mean
nice & easy to express
and
nice & easy to find
and
nice & easy to use
and
nice & easy to remember;
our consideration of triangles
has mostly to do with their shape
& little to do with their size

☐ ten notable triangles appear in the following rhombi when diagonals are drawn:

- a rhombus with all vertex angles of 90^0

= a square

- a rhombus with adjacent vertex angles of 60^0 and 120^o
- a rhombus with adjacent vertex angles of 45^0 and 135^o
- a rhombus with adjacent vertex angles of 30^0 and 150^o

note that simple fractions of a right angle
give the above acute angles
viz

$$\frac{1}{2} \times 90^o = 45^o$$

$$\frac{1}{3} \times 90^o = 30^o$$

$$\frac{2}{3} \times 90^o = 60^o$$

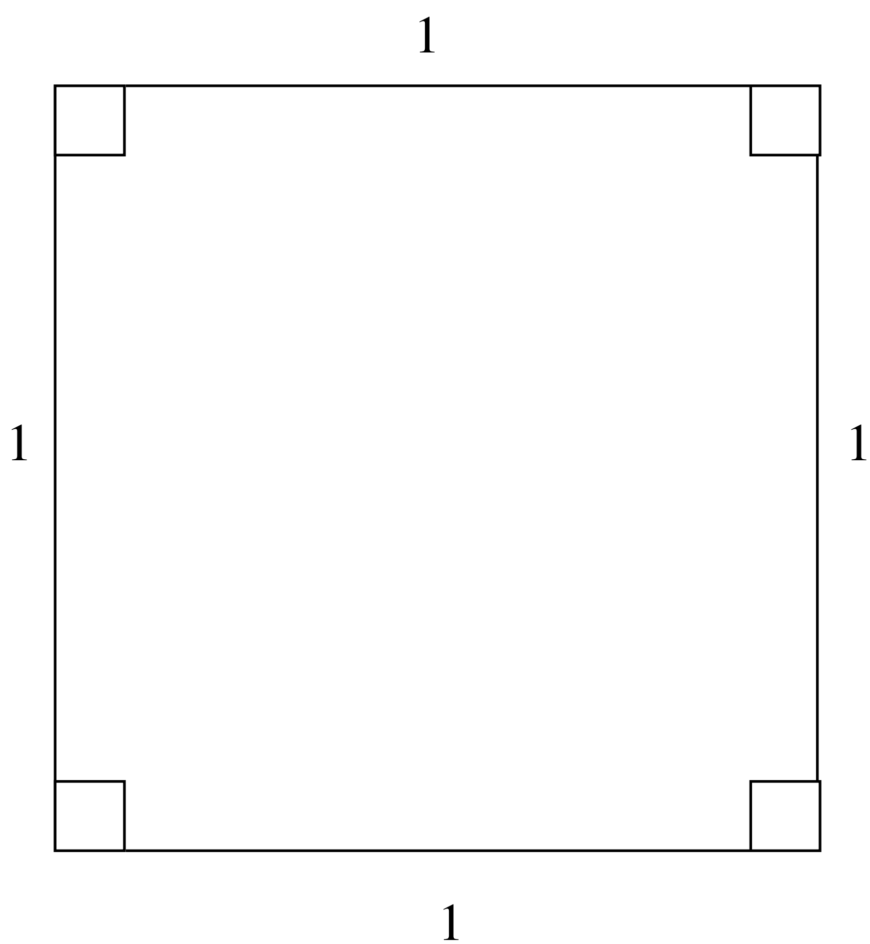
&

their supplements
give the above obtuse angles

• an isosceles right triangle
= a 45 - 45 - 90 degree triangle
= a one - one - two angle triangle
has its opposite sides in the ratio
$1 : 1 : \sqrt{2}$

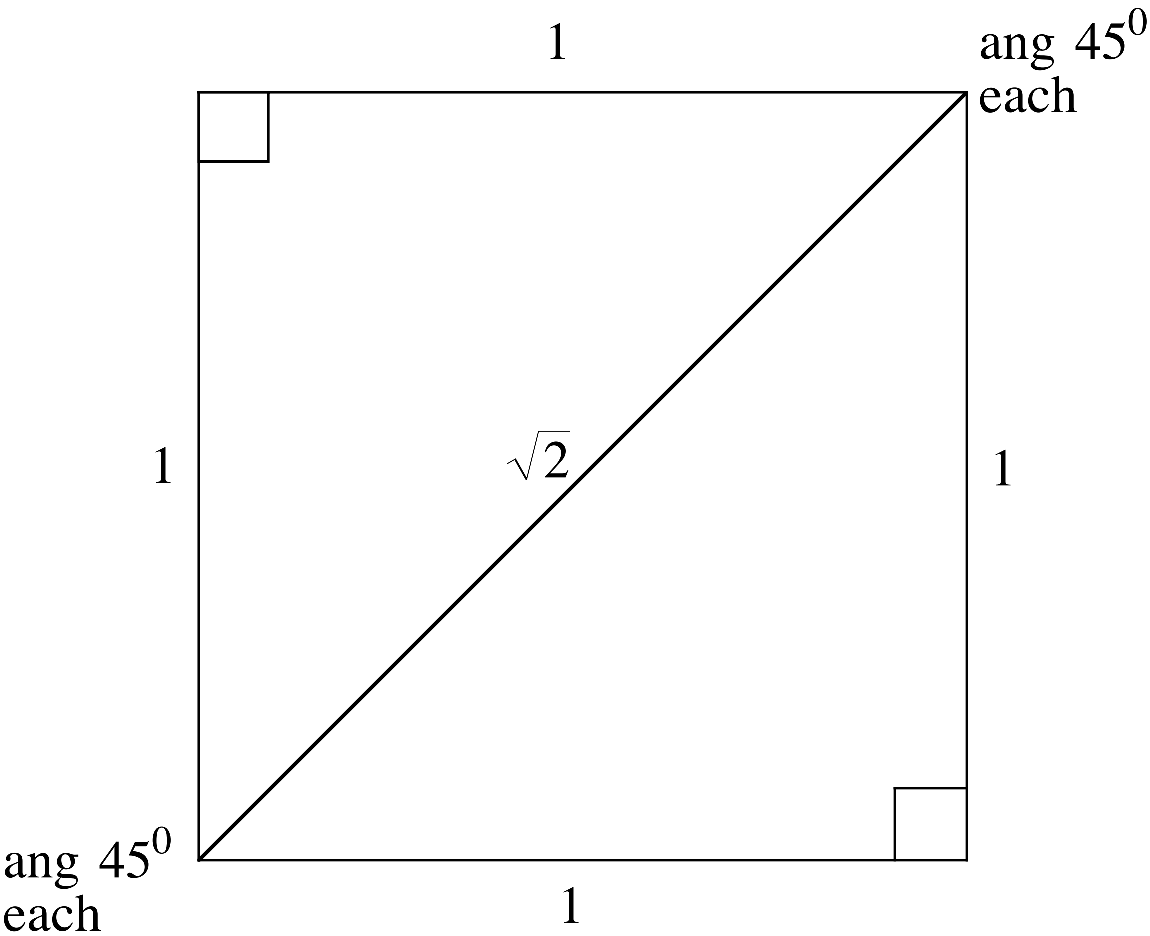
• to construct
an isosceles right triangle:
draw a square
&
draw a diagonal

• from
square
to
isosceles right triangle
in
three pictures & two steps



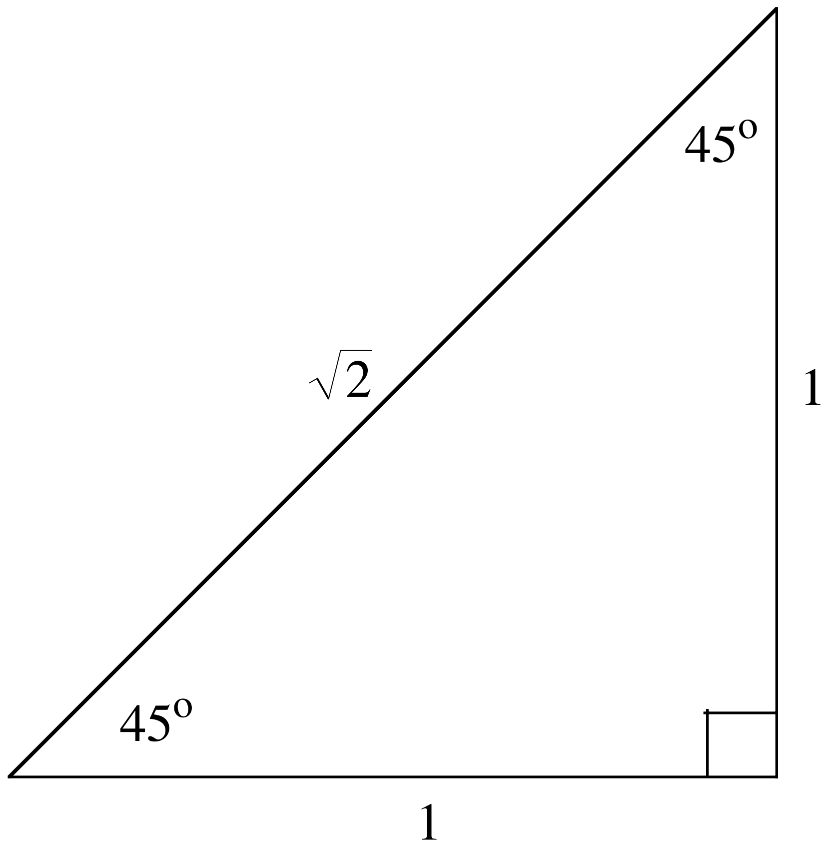
square

GG84-6



square with diagonal

GG84-7



isosceles right triangle

GG84-8

- the six basic trig fcns of 45^o

$$\sin 45^o = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^o = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^o = 1$$

$$\cot 45^o = 1$$

$$\sec 45^o = \sqrt{2}$$

$$\csc 45^o = \sqrt{2}$$

- an equilateral triangle

= an equiangular triangle

= a 60 - 60 - 60 degree triangle

= a one - one - one angle triangle

has its opposite sides in the ratio

1 : 1 : 1

- to construct

an equilateral triangle:

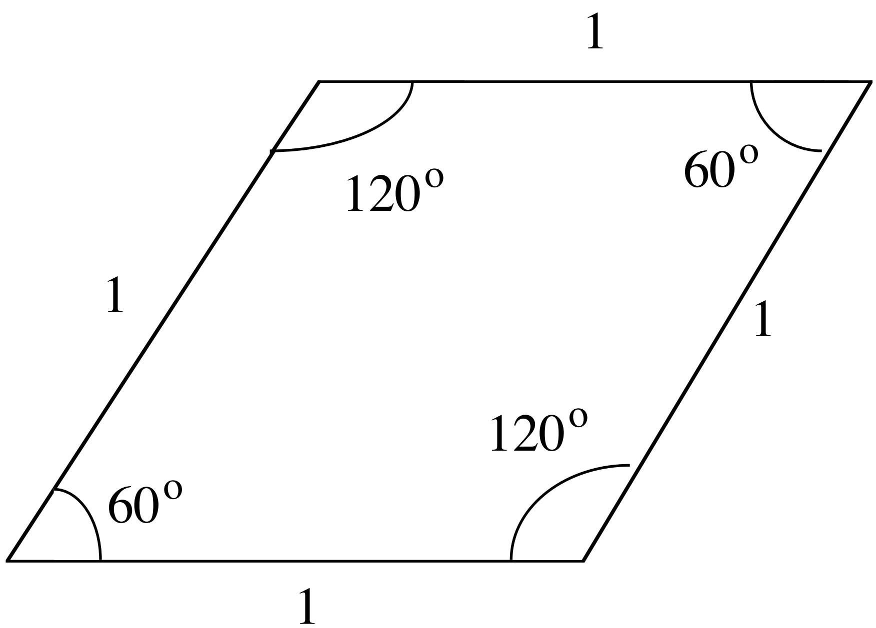
draw a rhombus

with vertex angles 60^{o} and 120^{o}

&

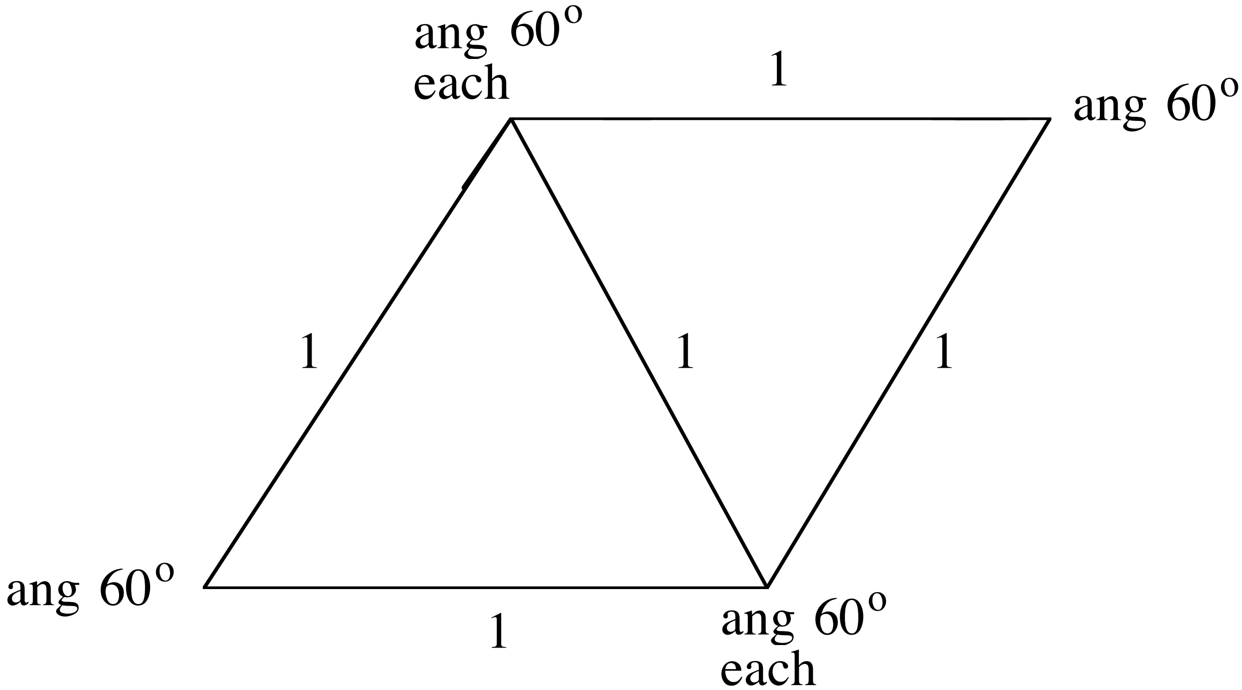
draw the shorter diagonal

• from
rhombus
to
equilateral triangle
in
three pictures & two steps



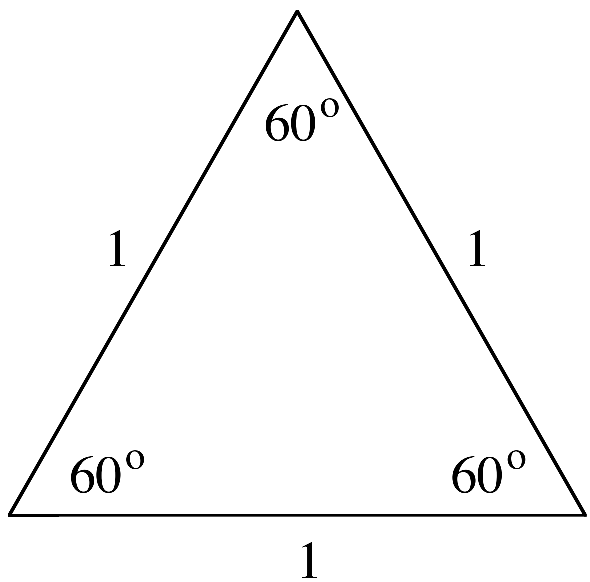
rhombus

GG84-11



rhombus with shorter diagonal

GG84-12



equilateral triangle

GG84-13

- an isosceles trine triangle

= a $30 - 30 - 120$ degree triangle

= a one - one - four angle triangle

has its opposite sides in the ratio

$1 : 1 : \sqrt{3}$

- to construct

an isosceles trine triangle:

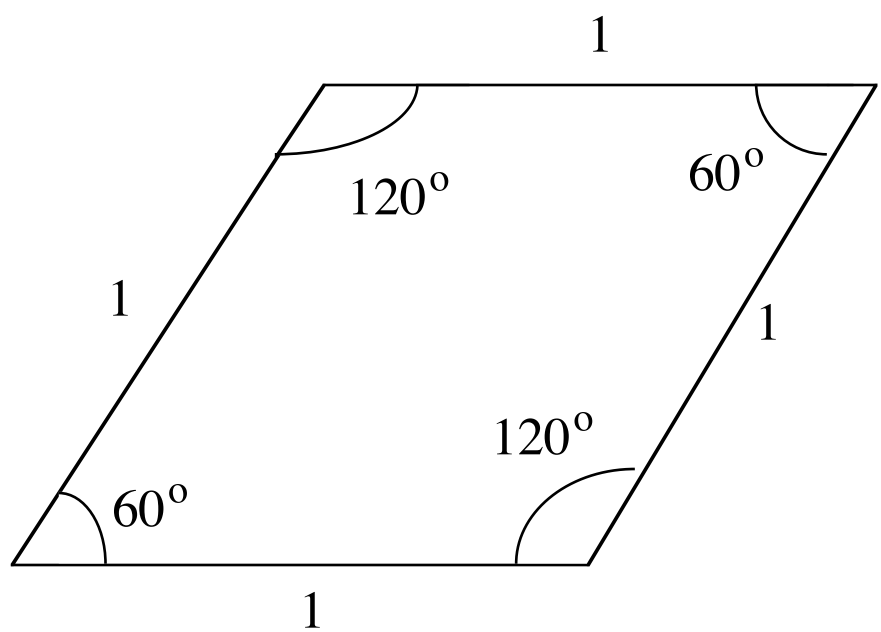
draw a rhombus

with vertex angles 60^{o} and 120^{o}

&

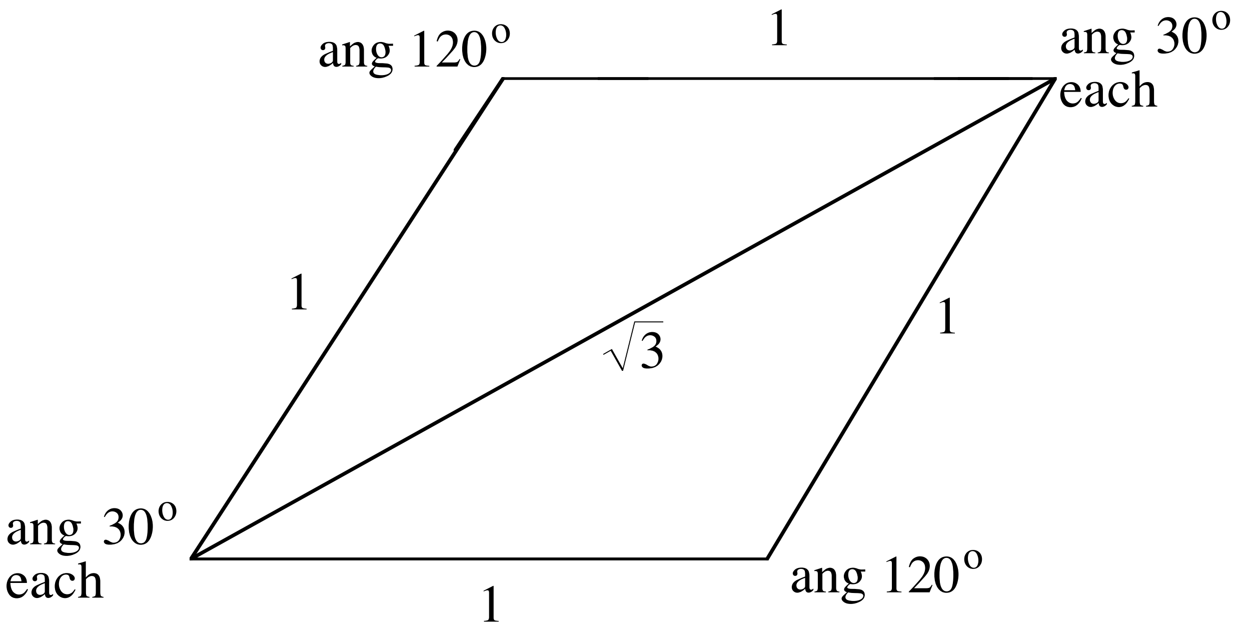
draw the longer diagonal

• from
rhombus
to
isosceles trine triangle
in
three pictures & two steps



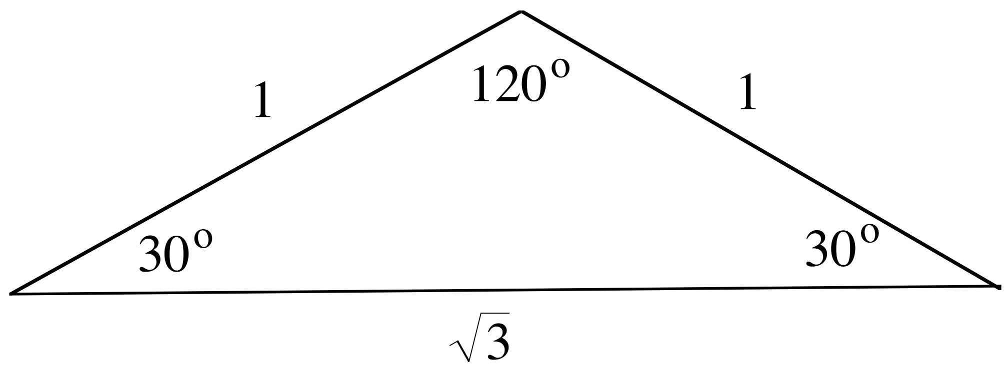
rhombus

GG84-15



rhombus with longer diagonal

GG84-16



isosceles trine trangle

GG84-17

- a sextile right triangle

= a $30 - 60 - 90$ degree triangle

= a one - two - three angle triangle

has its opposite sides in the ratio

$1 : \sqrt{3} : 2$

- to construct

a sextile right triangle:

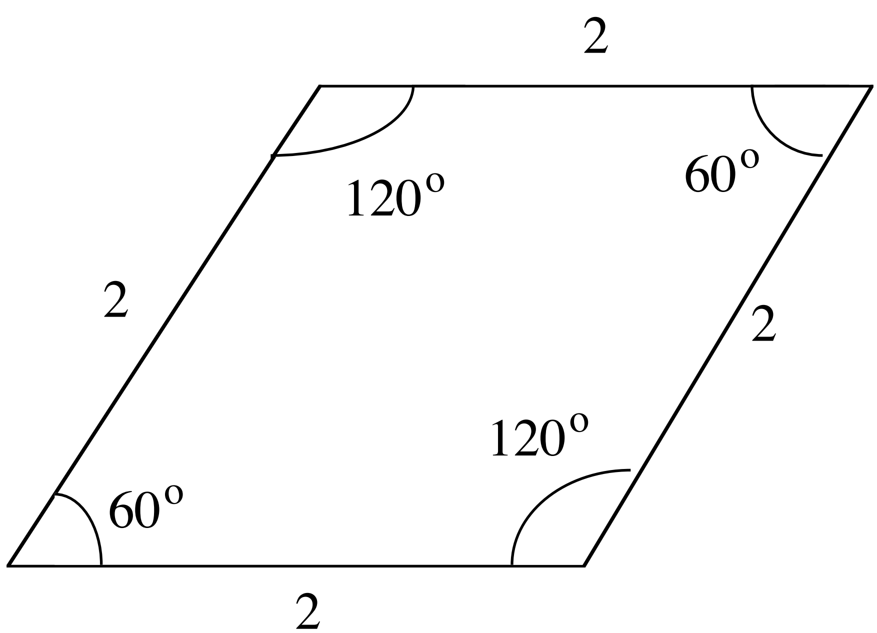
draw a rhombus

with vertex angles 60^{o} and 120^{o}

&

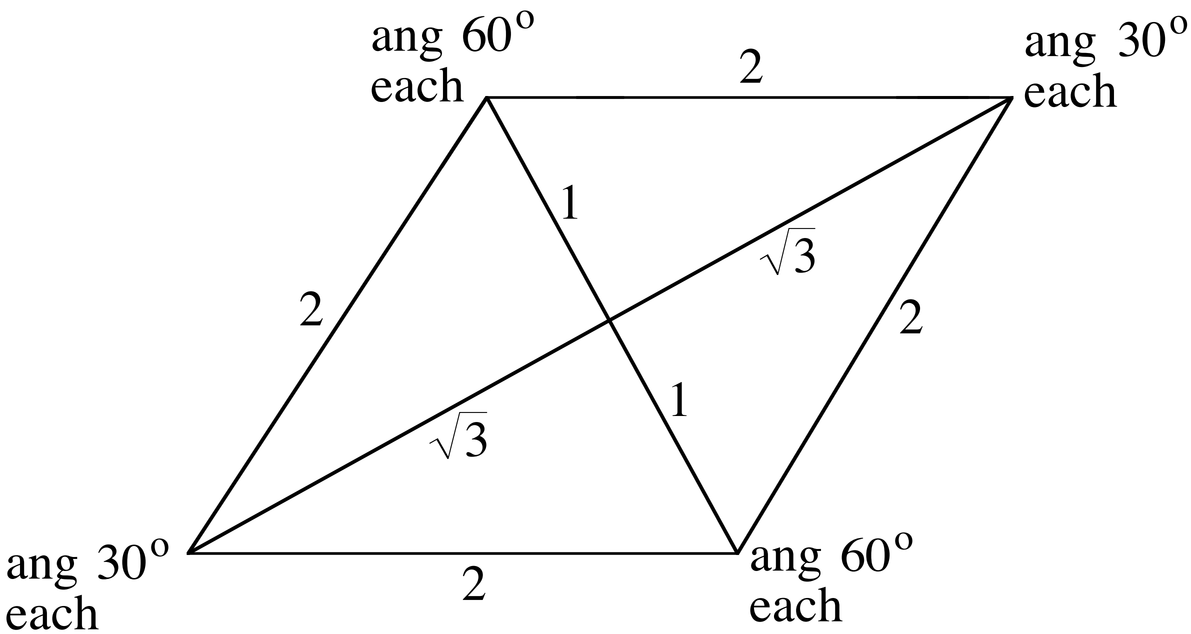
draw both diagonals

• from
rhombus
to
sextile right triangle
in
three pictures & two steps



rhombus

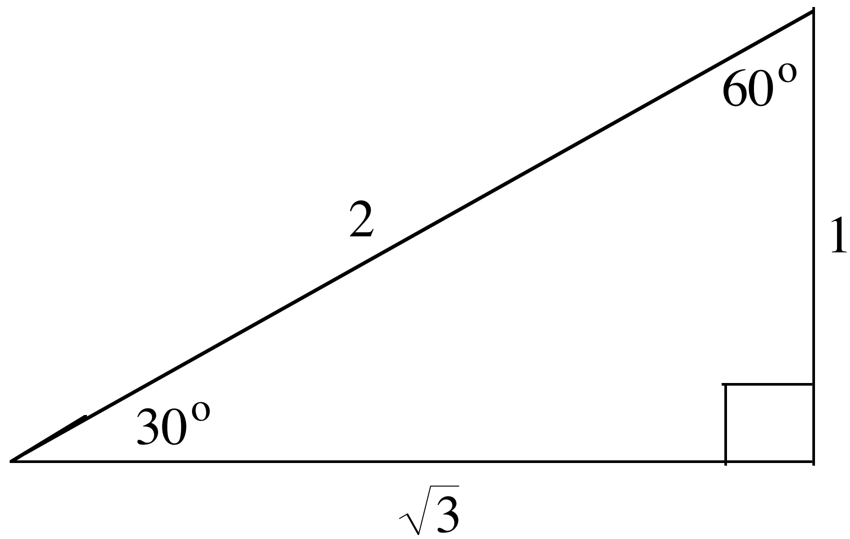
GG84-19

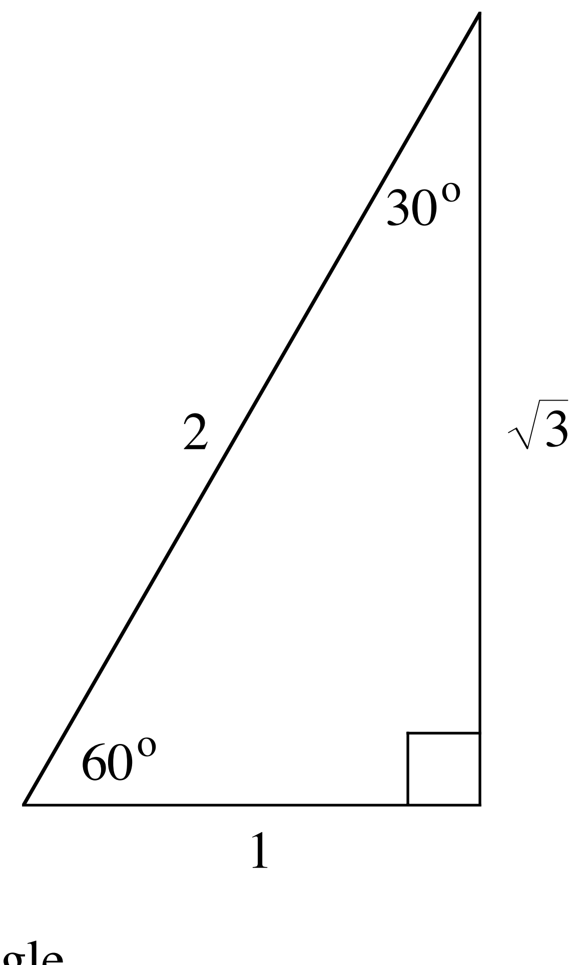


diagonals ⊥ bisectors of each other

rhombus with both diagonals

GG84-20





sextile right triangle

GG84-21

- the six basic trig fcns

of the complementary angles 30^o and 60^o

$$\sin 30^o = \cos 60^o = \frac{1}{2}$$

$$\cos 30^o = \sin 60^0 = \frac{\sqrt{3}}{2}$$

$$\tan 30^o = \cot 60^o = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot 30^o = \tan 60^0 = \sqrt{3}$$

$$\sec 30^o = \csc 60^o = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\csc 30^o = \sec 60^o = 2$$

- an isosceles semiright triangle

= a 45 - 67.5 - 67.5 degree triangle

= a two - three - three angle triangle

has its opposite sides in the ratio

$\sqrt{2-\sqrt{2}}$: 1 : 1

- to construct

an isosceles semiright triangle:

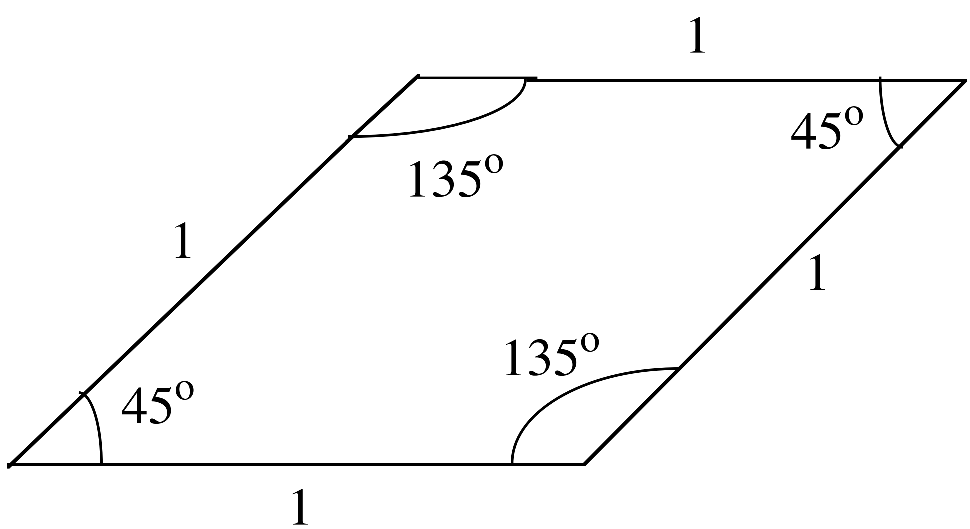
draw a rhombus

with vertex angles 45° and 135°

&

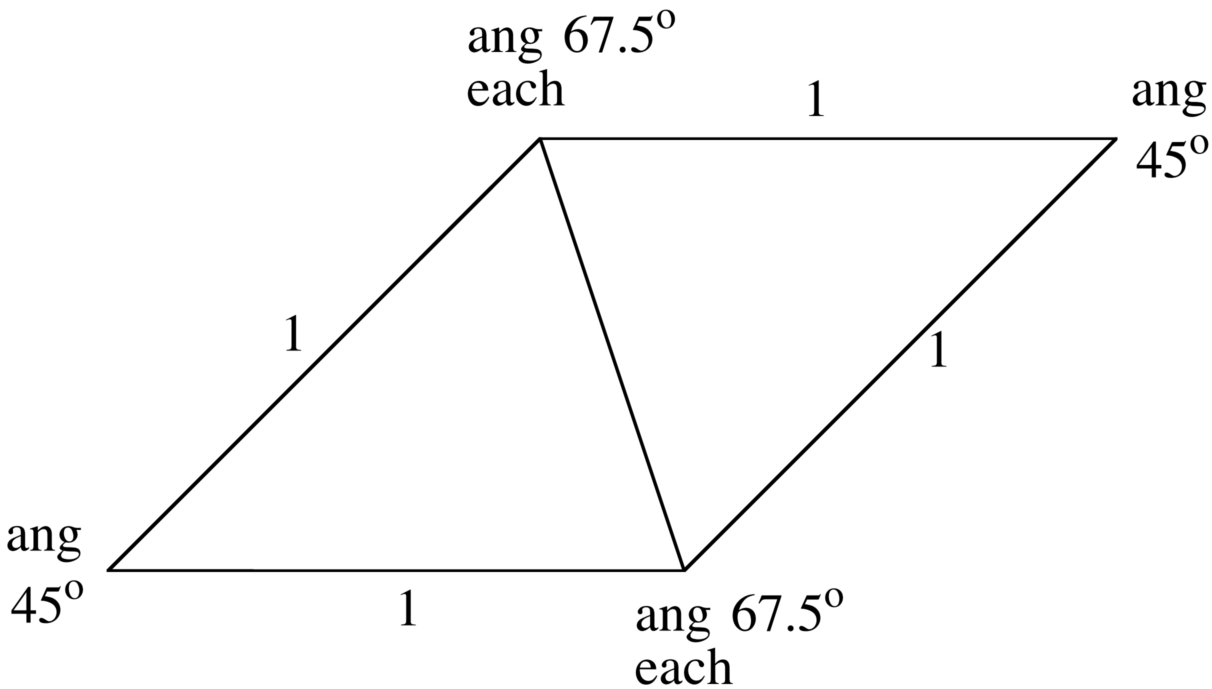
draw the shorter diagonal

• from
rhombus
to
isosceles semiright triangle
in
three pictures & two steps



rhombus

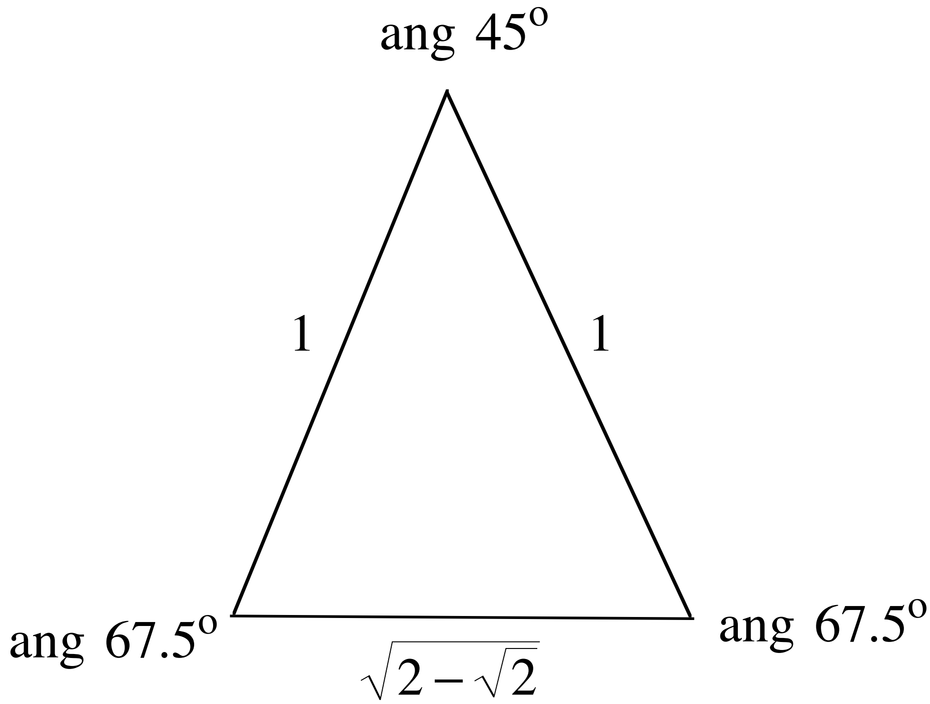
GG84-24



short diag = $\sqrt{2-\sqrt{2}}$

rhombus with shorter diagonal

GG84-25



isosceles semiright triangle

GG84-26

- an isosceles sesquiright triangle

= a 22.5 – 22.5 – 135 degree triangle

= a one - one - six angle triangle

has its opposite sides in the ratio

$1 : 1 : \sqrt{2+\sqrt{2}}$

- to construct

an isosceles sesquiright triangle:

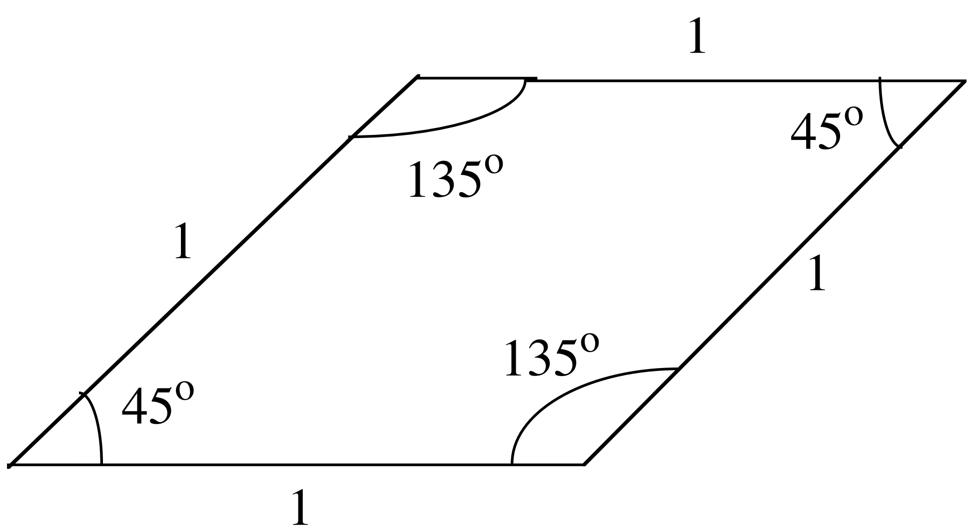
draw a rhombus

with vertex angles 45^{o} and 135^{o}

&

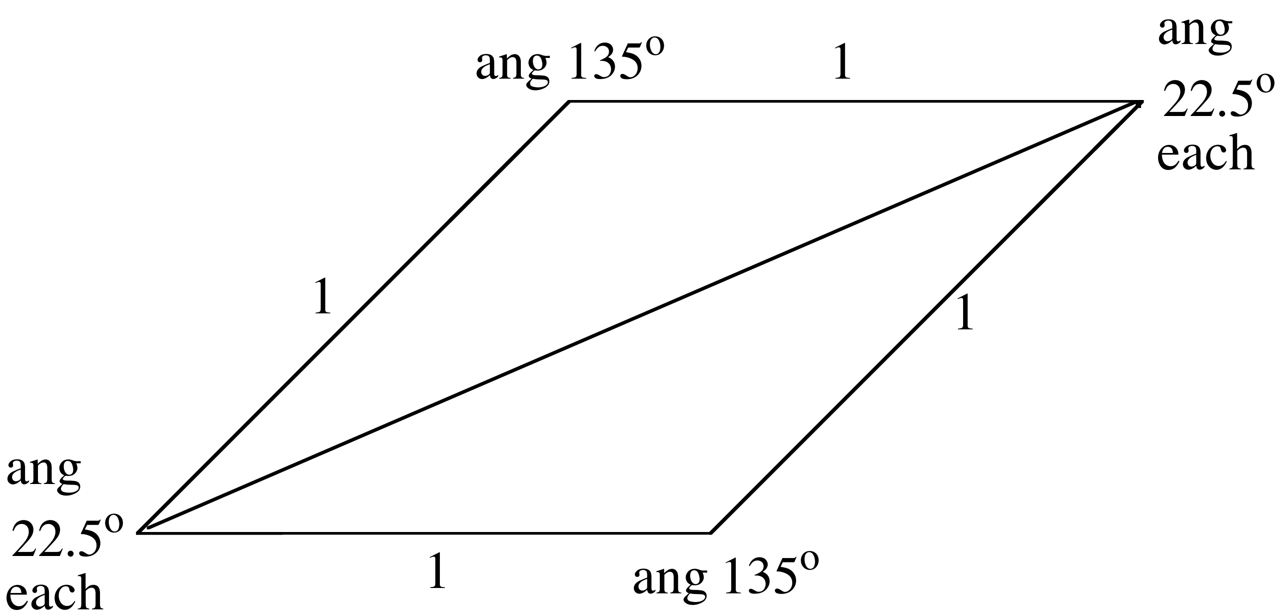
draw the longer diagonal

• from
rhombus
to
isosceles sesquiright triangle
in
three pictures & two steps



rhombus

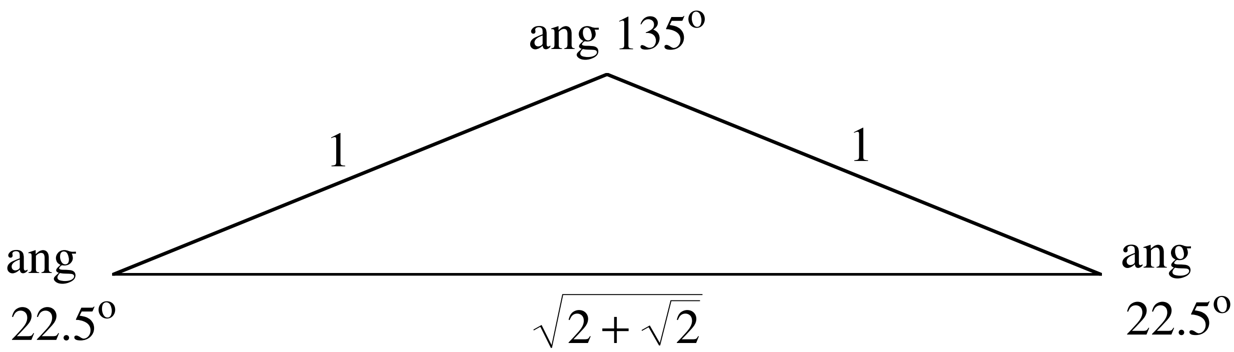
GG84-28



long diag $= \sqrt{2+\sqrt{2}}$

rhombus with longer diagonal

GG84-29



isosceles sesquiright triangle

- a quarteright right triangle

= a 22.5 - 67.5 - 90 degree triangle

= a one - three - four angle triangle

has its opposite sides in the ratio

$\sqrt{2-\sqrt{2}} : \sqrt{2+\sqrt{2}} : 2$

- to construct

a quarteright right triangle:

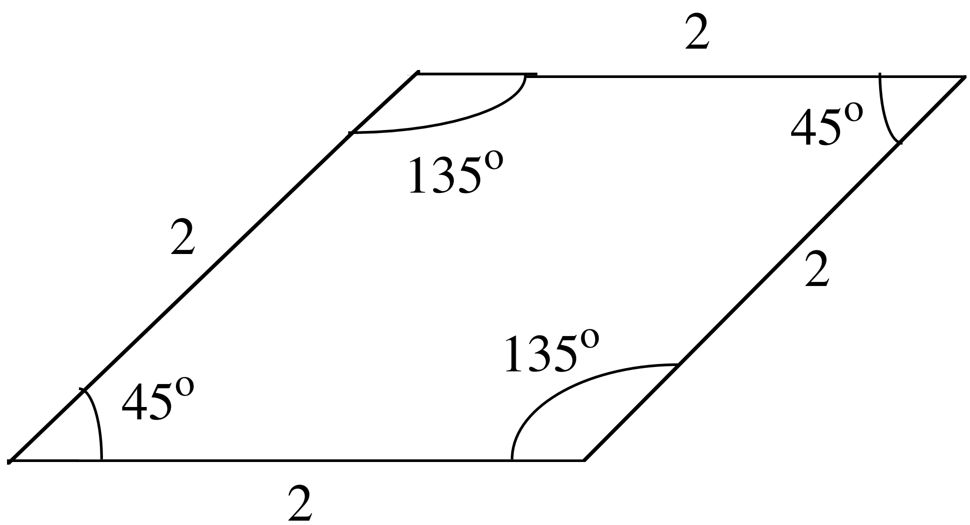
draw a rhombus

with vertex angles 45^{o} and 135^{o}

&

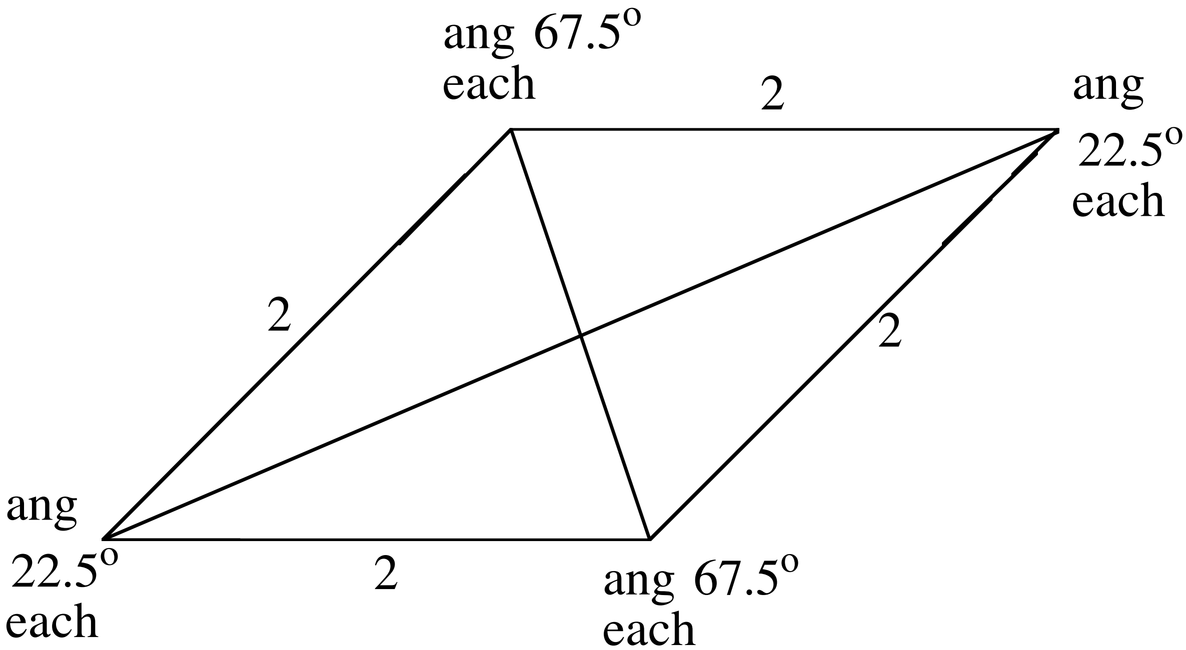
draw both diagonals

• from
rhombus
to
quarteright right triangle
in
three pictures & two steps



rhombus

GG84-32



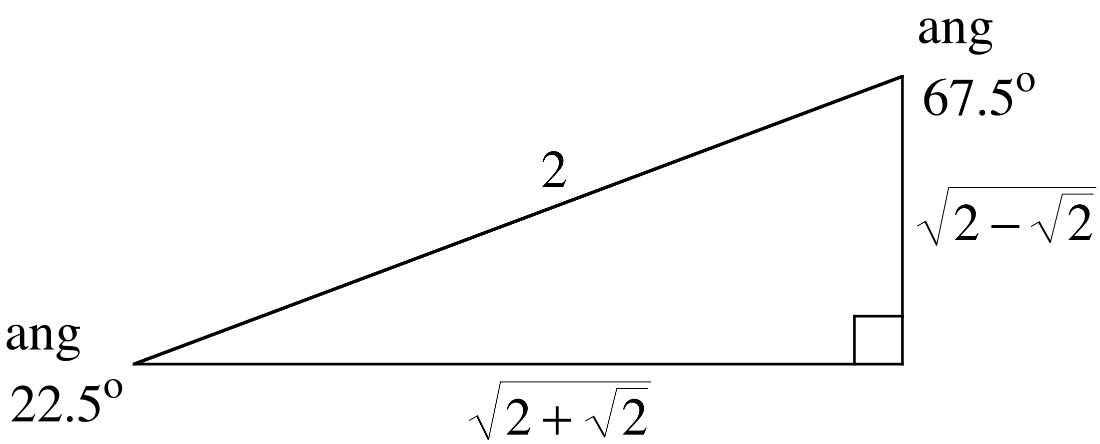
diagonals ⊥ bisectors of each other

short halfdiag = $\sqrt{2-\sqrt{2}}$

long halfdiag = $\sqrt{2+\sqrt{2}}$

rhombus with both diagonals

GG84-33



quarteright right triangle

GG84-34

- the six basic trig fcns

of the complementary angles 22.5° and 67.5°

$$\sin 22.5^{\circ} = \cos 67.5^{\circ} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\cos 22.5^{\circ} = \sin 67.5^{0} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\tan 22.5^{\circ} = \cot 67.5^{\circ} = \sqrt{2}-1$$

$$\cot 22.5^{\circ} = \tan 67.5^{0} = \sqrt{2}+1$$

$$\sec 22.5^{\circ} = \csc 67.5^{\circ} = \sqrt{4-2\sqrt{2}}$$

$$\csc 22.5^{\circ} = \sec 67.5^{\circ} = \sqrt{4+2\sqrt{2}}$$

- an isosceles semisextile triangle

= a 30 - 75 - 75 degree triangle

= a two - five - five angle triangle

has its opposite sides in the ratio

$\sqrt{2-\sqrt{3}}$: 1 : 1

- to construct

an isosceles semisextile triangle:

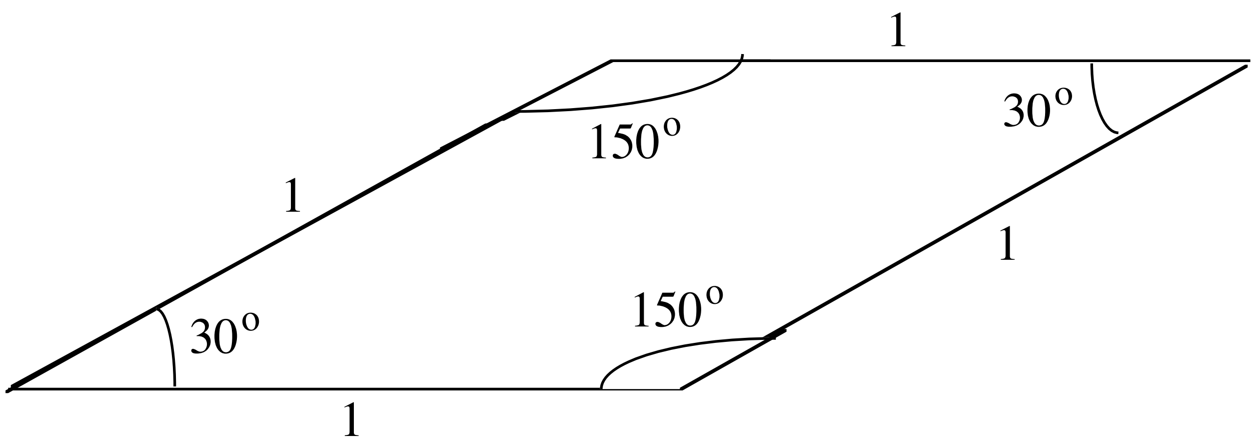
draw a rhombus

with vertex angles 30^{o} and 150^{o}

&

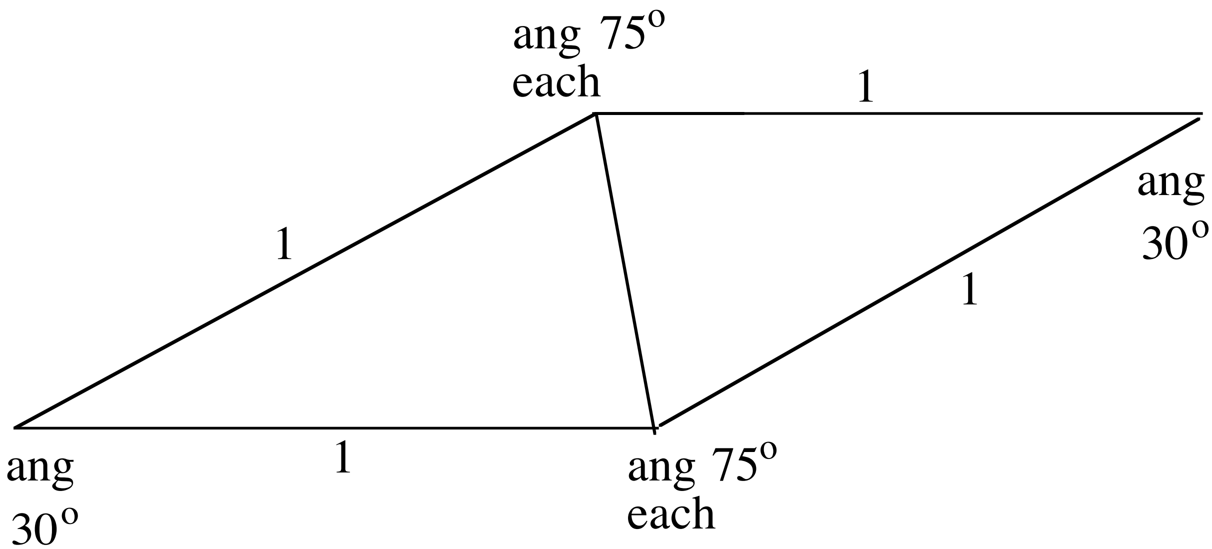
draw the shorter diagonal

• from
rhombus
to
isosceles semisextile triangle
in
three pictures & two steps



rhombus

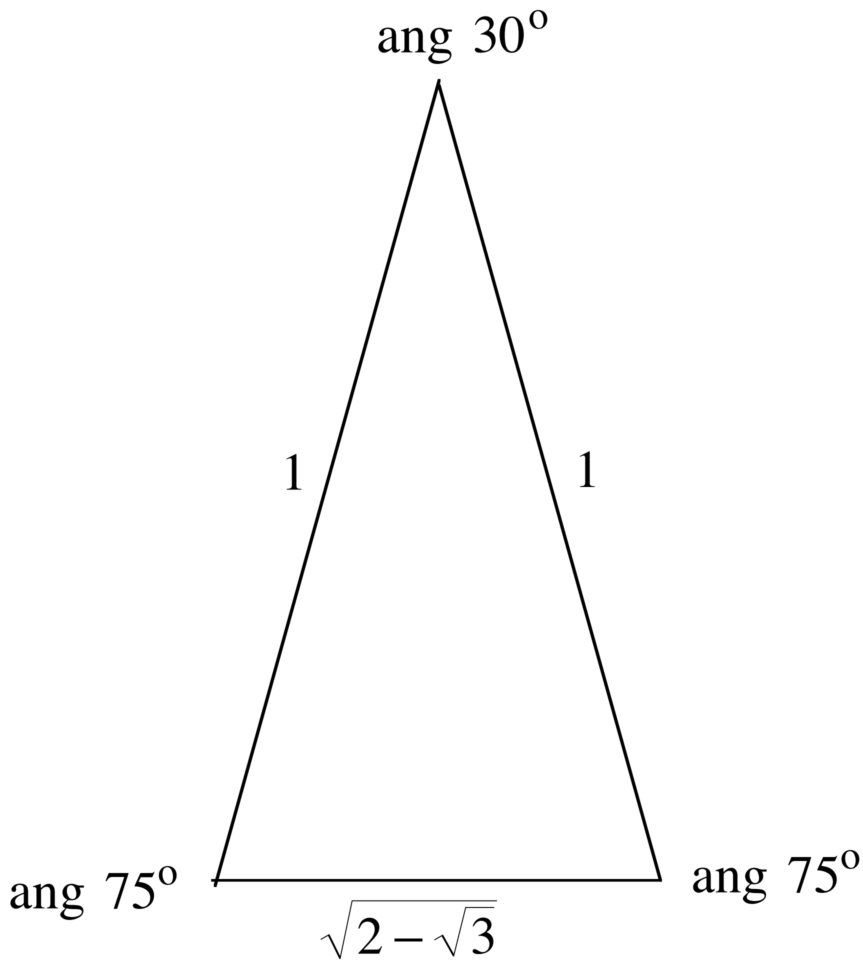
GG84-37



short diag = $\sqrt{2-\sqrt{3}}$

rhombus with shorter diagonal

GG84-38



isosceles semisextile triangle

GG84-39

- an isosceles sesquicental triangle

= a 15 - 15 - 150 degree triangle

= a one - one - ten angle triangle

has its opposite sides in the ratio

$1 : 1 : \sqrt{2+\sqrt{3}}$

- to construct

an isosceles sesquicental triangle:

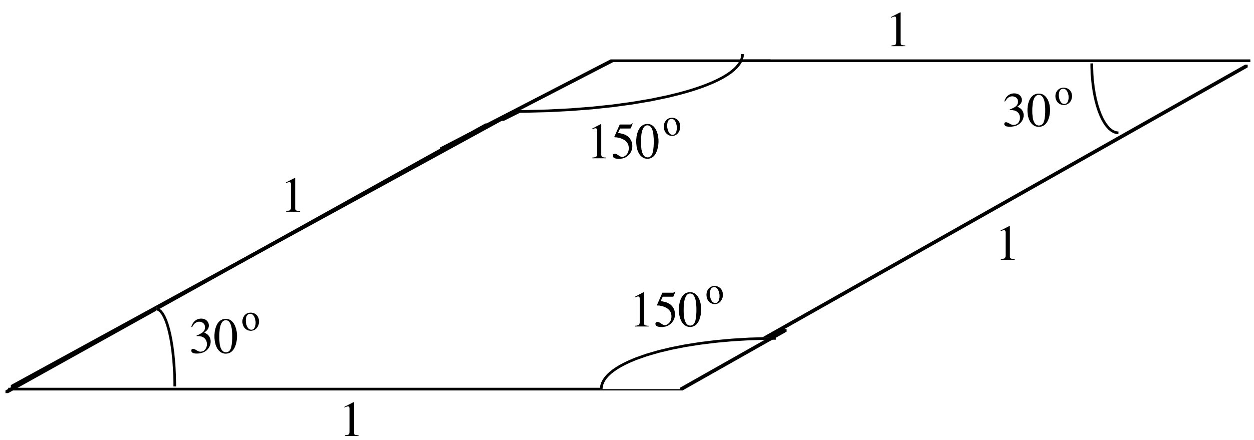
draw a rhombus

with vertex angles 30^{o} and 150^{o}

&

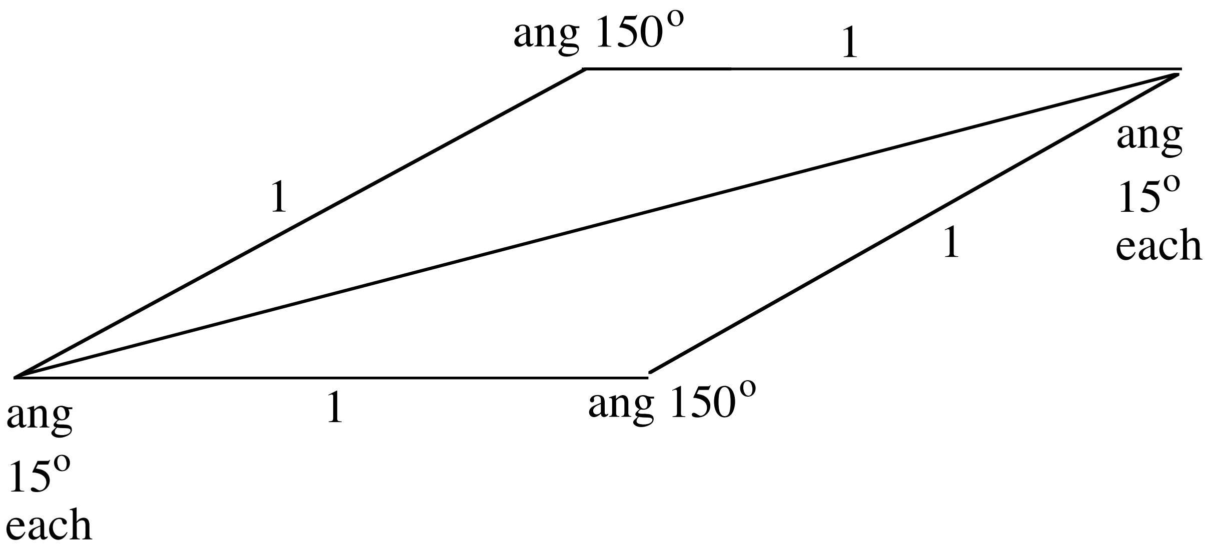
draw the longer diagonal

• from
rhombus
to
isosceles sesquicental triangle
in
three pictures & two steps



rhombus

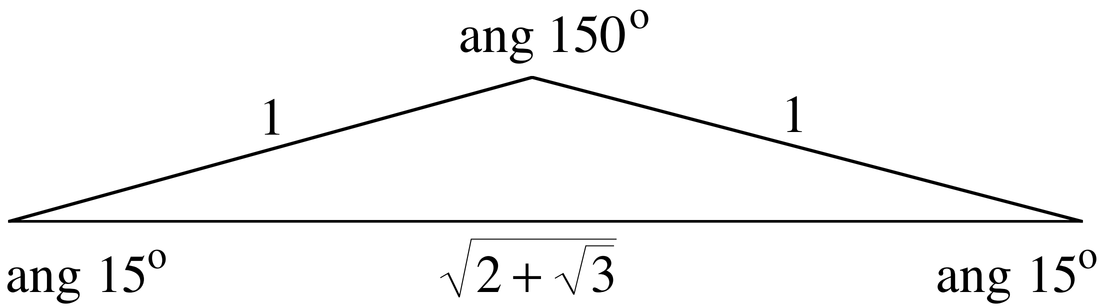
GG84-41



long diag = $\sqrt{2+\sqrt{3}}$

rhombus with longer diagonal

GG84-42



isosceles sesquicental triangle

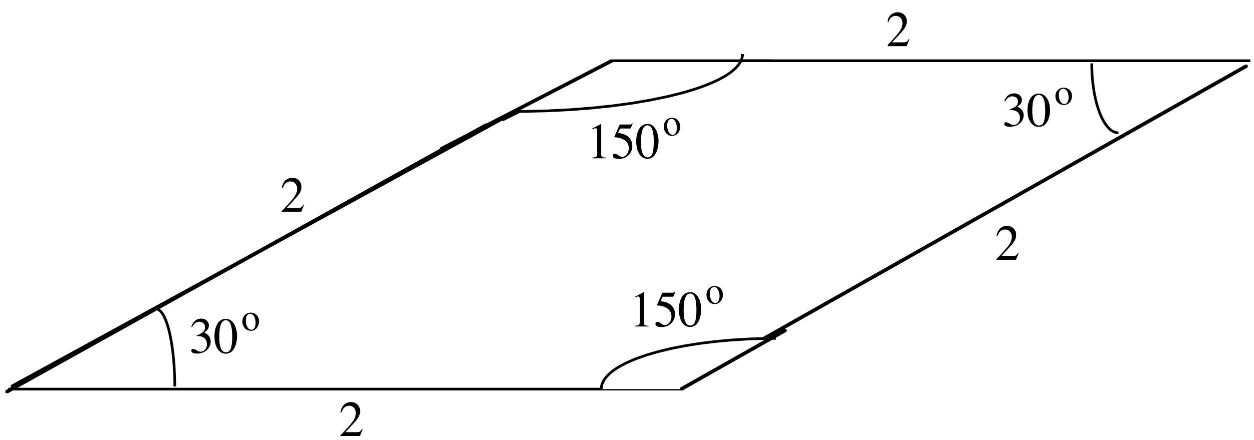
GG84-43

• a quartersextile right triangle
= a 15 - 75 - 90 degree triangle
= a one - five - six angle triangle
has its opposite sides in the ratio

$\sqrt{2-\sqrt{3}} : \sqrt{2+\sqrt{3}} : 2$

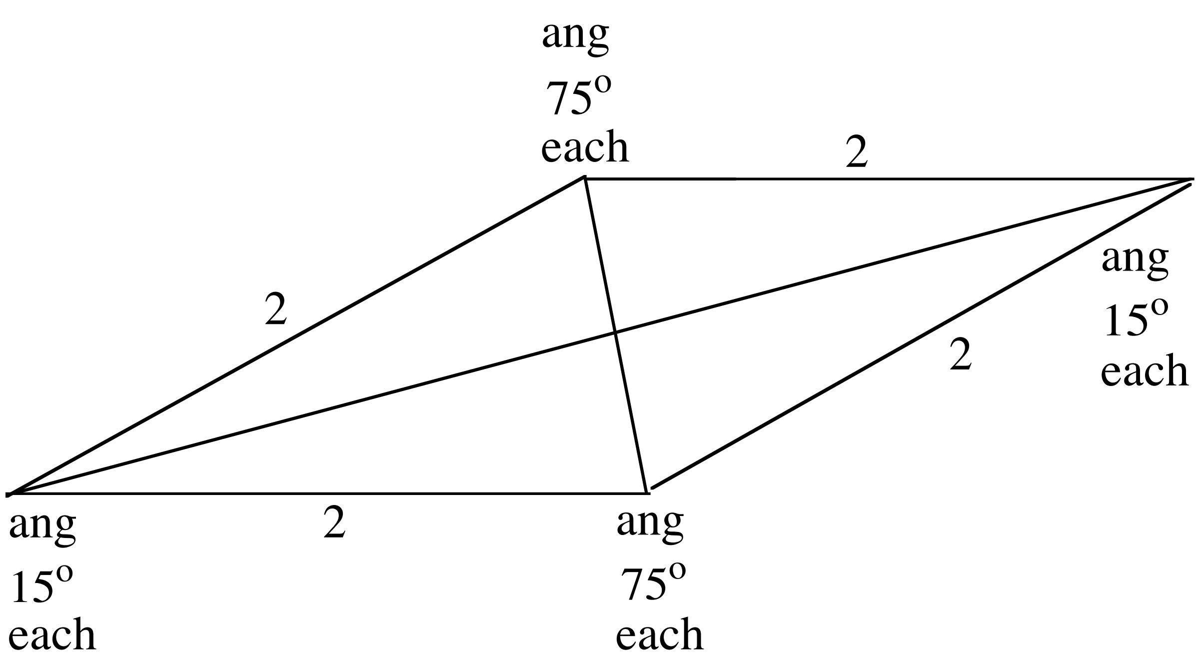
• to construct
a quartersextile right triangle:
draw a rhombus
with vertex angles 30^{o} and 150^{o}
&
draw both diagonals

• from
rhombus
to
quartersextile right triangle
in
three pictures & two steps



rhombus

GG84-45



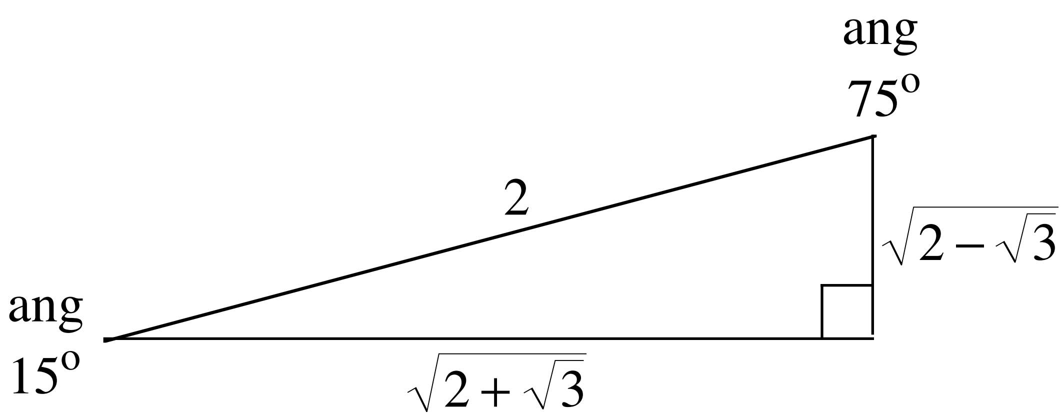
diagonals ⊥ bisectors of each other

short halfdiag = $\sqrt{2-\sqrt{3}}$

long halfdiag = $\sqrt{2+\sqrt{3}}$

rhombus with both diagonals

GG84-46



quartersextile right triangle

GG84-47

- the six basic trig fcns

of the complementary angles 15^{o} and 75^{o}

$$\sin 15^{o} \ = \ \cos 75^{o} \ = \ \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\cos 15^{o} \ = \ \sin 75^{0} \ = \ \frac{\sqrt{2+\sqrt{3}}}{2}$$

$$\tan 15^{o} \ = \ \cot 75^{o} \ = \ 7-4\sqrt{3}$$

$$\cot 15^{o} \ = \ \tan 75^{0} \ = \ 7+4\sqrt{3}$$

$$\sec 15^{o} \ = \ \csc 75^{o} \ = \ 2\sqrt{2-\sqrt{3}}$$

$$\csc 15^{o} \ = \ \sec 75^{o} \ = \ 2\sqrt{2+\sqrt{3}}$$

☐ the golden ratio

$=_{df}$ the positive real number x st

$$\frac{x+1}{x} = \frac{x}{1}$$

which has the geometric interpretation of

dividing a line segment of whole length $x+1$

into subsegments

of larger length x and of smaller length 1 st

their lengths satisfy the proportion

the whole is to the larger

as

the larger is to the smaller;

this gives the quadratic equation

$$x^2 - x - 1 = 0$$

with unique positive root

$$x = \frac{1+\sqrt{5}}{2} = 1.61803+$$

which is denoted by the lowercase Greek letter phi

φ

and which is called

the golden ratio

• note that the golden ratio

$$\varphi = \frac{1}{2}\left(1+\sqrt{5}\right) = 1.61803+ \approx 1.6 = \frac{8}{5} = 8 : 5$$

• a rectangle whose sides are in the ratio $\varphi : 1$
is called
a golden rectangle;
a golden rectangle is considered to be
esthetically pleasing
& was so recognized by the ancient Greeks

☐ how to compute cos 36^{o} exactly
using only trig & algebra

set $A = 36^{o}$
then

$$5A = 180^{o}$$

$$3A = 180^{o} - 2A$$

$$\cos 3A = -\cos 2A$$

$$\cos 3A + \cos 2A = 0$$

$$4\cos^{3} A - 3\cos A + 2\cos^{2} A - 1 = 0$$

$$4\cos^{3} A + 2\cos^{2} A - 3\cos A - 1 = 0$$

set $x = \cos A$
then

$$4x^{3} + 2x^{2} - 3x - 1 = 0$$

$$(x+1)(4x^{2} - 2x - 1) = 0$$

$$4x^{2} - 2x - 1 = 0$$

$$x = \frac{2 + \sqrt{20}}{8} = \frac{1 + \sqrt{5}}{4} = \frac{\varphi}{2}$$

$$\therefore \cos 36^{o} = \frac{1 + \sqrt{5}}{4} = \frac{\varphi}{2}$$

GG84-51

□ how to compute $\cos 36^{\circ}$ exactly
using a little bit of geometry

consider an isosceles triangle
with apex angle $= 36^{\circ}$
with each base angle $= 72^{\circ}$;
bisect a base angle
& consider how the bisector divides the opposite side;
take the segment with endpoint at the apex to be x
& the segment with endpoint at the base to be 1;
by similar triangles

$\frac{x+1}{x} = \frac{x}{1}$ which is the golden ratio proportion & thus

$x = \varphi$;

by the law of sines

$$\frac{\sin 36^{\circ}}{1} = \frac{\sin 72^{\circ}}{\varphi} = \frac{2\sin 36^{\circ}\cos 36^{\circ}}{\varphi} \quad \text{\& thus}$$

$$\cos 36^{\circ} = \frac{\varphi}{2} = \frac{1+\sqrt{5}}{4}$$

☐ an isosceles triangle
whose slant side is to the base as φ : 1
is called
a golden triangle;
the apex angle of a golden triangle is 36^o
& each base angle is 72^o

• a golden triangle
= a 36 - 72 - 72 degree triangle
= a one - two - two angle triangle
has its opposite sides in the ratio
1 : φ : φ

- consider a golden triangle

with base 1

& slant sides $\varphi = \frac{1}{2}\left(1+\sqrt{5}\right)$

then

the altitude to the base

$$= \frac{1}{2}\sqrt{4\varphi+3} = \frac{1}{2}\sqrt{5+2\sqrt{5}}$$

&

the altitudes to the slant sides

$$= \frac{1}{2}\sqrt{\varphi+2} = \frac{1}{4}\sqrt{10+2\sqrt{5}}$$

• the following two triangles
appear in a golden triangle
when the altitudes are drawn

* a 36 - 54 - 90 degree triangle
= a two - three - five angle triangle
has its opposite sides in the ratio
$\sqrt{3-\varphi} : \varphi : 2$

* a 18 - 72 - 90 degree triangle
= a one - four - five angle triangle
has its opposite sides in the ratio
$1 : \sqrt{4\varphi+3} : 2\varphi$

- the six basic trig fcns

of the complementary angles 36^{o} and 54^{o}

$$\sin 36^{\mathrm{o}} \;=\; \cos 54^{\mathrm{o}} \;=\; \frac{\sqrt{3-\varphi}}{2}$$

$$\cos 36^{\mathrm{o}} \;=\; \sin 54^{0} \;=\; \frac{\varphi}{2}$$

$$\tan 36^{\mathrm{o}} \;=\; \cot 54^{\mathrm{o}} \;=\; \frac{\sqrt{3-\varphi}}{\varphi}$$

$$\cot 36^{\mathrm{o}} \;=\; \tan 54^{0} \;=\; \frac{\varphi}{\sqrt{3-\varphi}}$$

$$\sec 36^{\mathrm{o}} \;=\; \csc 54^{\mathrm{o}} \;=\; \frac{2}{\varphi}$$

$$\csc 36^{\mathrm{o}} \;=\; \sec 54^{\mathrm{o}} \;=\; \frac{2}{\sqrt{3-\varphi}}$$

- the six basic trig fcns

of the complementary angles 18^o and 72^o

$$\sin 18^o \;=\; \cos 72^o \;=\; \frac{1}{2\varphi}$$

$$\cos 18^o \;=\; \sin 72^0 \;=\; \frac{\sqrt{4\varphi+3}}{2\varphi}$$

$$\tan 18^o \;=\; \cot 72^o \;=\; \frac{1}{\sqrt{4\varphi+3}}$$

$$\cot 18^o \;=\; \tan 72^0 \;=\; \sqrt{4\varphi+3}$$

$$\sec 18^o \;=\; \csc 72^o \;=\; \frac{2\varphi}{\sqrt{4\varphi+3}}$$

$$\csc 18^o \;=\; \sec 72^o \;=\; 2\varphi$$

☐ consider

an isosceles triangle

with each slantside $= a$

with base $= b$

with apex angle $= \alpha$

with each base angle $= \beta$;

then

by the law of cosines

$$\cos\alpha = \frac{2a^2 - b^2}{2a^2}$$

from which the trig fcns of α and β

can be computed

viz

$$\sin\alpha = \frac{b\sqrt{4a^2 - b^2}}{2a^2}$$

$$\cos\alpha = \frac{2a^2 - b^2}{2a^2}$$

$$\tan\alpha = \frac{b\sqrt{4a^2 - b^2}}{2a^2 - b^2}$$

$$\cot\alpha = \frac{2a^2 - b^2}{b\sqrt{4a^2 - b^2}}$$

$$\sec\alpha = \frac{2a^2}{2a^2 - b^2}$$

$$\csc\alpha = \frac{2a^2}{b\sqrt{4a^2 - b^2}}$$

$$\sin\beta = \frac{\sqrt{4a^2 - b^2}}{2a}$$

$$\cos\beta = \frac{b}{2a}$$

$$\tan\beta = \frac{\sqrt{4a^2 - b^2}}{b}$$

$$\cot\beta = \frac{b}{\sqrt{4a^2 - b^2}}$$

$$\sec\beta = \frac{2a}{b}$$

$$\csc\beta = \frac{2a}{\sqrt{4a^2 - b^2}}$$

□ an interesting scalene triangle
that is not a right triangle

• the angles A, B, C of any triangle ABC
satisfy the identity
$\tan A + \tan B + \tan C = \tan A \tan B \tan C$
which expresses the fact that
for any triangle
the sum of the tangents of the angles
equals
their product;
a striking triple of numbers
such that their sum equals their product
is
1, 2, 3

- the tangent - one - two - three triangle

has

tangents of angles equal to

1, 2, 3

&

angles

$\tan^{-1} 1 = 45^{o}$

$\tan^{-1} 2 \approx 63^{o}$

$\tan^{-1} 3 \approx 72^{o}$

&

angle sines and sides in the ratio

$$\frac{1}{\sqrt{2}} : \frac{2}{\sqrt{5}} : \frac{3}{\sqrt{10}}$$

□ three triangles in a golden rectangle

- consider a golden rectangle

with base φ & height 1

- drawing one diagonal produces

a right triangle

with legs 1 and φ & hypotenuse $\sqrt{\varphi+2}$

and with acute angles whose tangents are

φ & $\frac{1}{\varphi}$

- drawing two diagonals produces

two isosceles triangles,

one acute - angled and one obtuse - angled;

the acute - angled triangle has sides 1, $\frac{1}{2}\sqrt{\varphi+2}$, $\frac{1}{2}\sqrt{\varphi+2}$

whose opposite angles have tangents 2, φ, φ;

the obtuse - angled triangle has sides φ, $\frac{1}{2}\sqrt{\varphi+2}$, $\frac{1}{2}\sqrt{\varphi+2}$

whose opposite angles have tangents -2, $\frac{1}{\varphi}$, $\frac{1}{\varphi}$