Deradicalizing Radicals

#82 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
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Infinite Vistas Press
PVD RI
2003

GG82-1 (48)

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□ we consider certain forms of
finite & infinite nested pth powers;
all considerations
are assumed to be about real numbers
&
are assumed to stay in the real field

□ we consider a special form of nested pth powers as defined below

- the basic iterated step:

replace something by
something plus a coefficient times a pth power

- start with

$$a_0 = x_0$$

- replace x_0 by $x_0 + c_1 x_1^p$ to get

$$a_1 = x_0 + c_1 x_1^p$$

- replace x_1 by $x_1 + c_2 x_2^p$ to get

$$a_2 = x_0 + c_1\left(x_1 + c_2 x_2^p\right)^p$$

- replace x_2 by $x_2 + c_3 x_3^p$ to get

$$a_3 = x_0 + c_1\left(x_1 + c_2\left(x_2 + c_3 x_3^p\right)^p\right)^p$$

etc

- if the process is ended in finitely many steps,

then a finite nested pth power is obtained;

if the process continues to

an infinite sequence of the a' s,

then an infinite nested pth power is obtained

and

the limit of the a' s if it exists

viz

$\lim_{k \to \infty} a_k$ (wh k $\in$ nonneg int var) iie

is denoted

$$x_0 + c_1\left(x_1 + c_2\left(x_2 + c_3(x_3 + \cdots)^p\right)^p \right)^p$$

□ the above notion of nested pth powers
subsumes the notions of
series & continued fraction

• if p & the c's are all = 1,
then the a's are
the partial sums of the series
$x_0 + x_1 + x_2 + x_3 + \cdots$
viz
$a_0 = x_0$
$a_1 = x_0 + x_1$
$a_2 = x_0 + x_1 + x_2$
$a_3 = x_0 + x_1 + x_2 + x_3$
etc

• if $p = -1$,

then the a' s are

the convergents of the continued fraction

$$x_0 + \cfrac{c_1}{x_1 + \cfrac{c_2}{x_2 + \cfrac{c_3}{x_3 + \cdots}}}$$

viz

$$a_0 = x_0$$

$$a_1 = x_0 + \frac{c_1}{x_1}$$

$$a_2 = x_0 + \cfrac{c_1}{x_1 + \cfrac{c_2}{x_2}}$$

$$a_3 = x_0 + \cfrac{c_1}{x_1 + \cfrac{c_2}{x_2 + \cfrac{c_3}{x_3}}}$$

etc

□ Herschfeld's Convergence Theorem

let

- $x_n \in$ nonneg real nr for n $\in$ nonneg int
- p $\in$ real nr st $0 < p < 1$
- define

$a_0 = x_0$

$a_1 = x_0 + x_1{}^p$

$a_2 = x_0 + \left(x_1 + x_2{}^p\right)$

$a_3 = x_0 + \left(x_1 + \left(x_2 + x_3{}^p\right)^p\right)$

etc

then

- $\exists \lim_{n\to\infty} a_n < \infty$

iff

- $\left\{(x_n)^{p^n} \middle| n \in \text{nonneg int}\right\}$ is bounded

D. quadratically constructible real numbers

- let r $\in$ real nr

then

- r is quadratically constructible

$=_{df}$ r is expressible ito
integers and only finitely many applications of
the operations of
addition, subtraction, multiplication, division,
& the extraction of the square root
of a nonnegative real number
to produce a unique nonnegative real number,
the process thus always staying in the real field

☐ some simple algebraic identities
for the denesting of nested square root expressions
assuming all letters stand for positive real numbers
& some obvious inequalities if a minus sign appears

- $\sqrt{\left(a^2+b^2c\right)+2ab\sqrt{c}} = a+b\sqrt{c}$

- $\sqrt{\left(a^2+b^2c\right)-2ab\sqrt{c}} = a-b\sqrt{c}$

- $\sqrt{(a+b)+2\sqrt{ab}} = \sqrt{a}+\sqrt{b}$

- $\sqrt{(a+b)-2\sqrt{ab}} = \sqrt{a}-\sqrt{b}$

- $\sqrt{(a+b+c)+2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc}} = \sqrt{a}+\sqrt{b}+\sqrt{c}$

- $\sqrt{(a+b+c)+2\sqrt{ab}-2\sqrt{ac}-2\sqrt{bc}} = \sqrt{a}+\sqrt{b}-\sqrt{c}$

- $\sqrt{(a+b+c)-2\sqrt{ab}-2\sqrt{ac}+2\sqrt{bc}} = \sqrt{a}-\sqrt{b}-\sqrt{c}$

eg

- $\sqrt{101+36\sqrt{5}} = 9+2\sqrt{5}$
- $\sqrt{101-36\sqrt{5}} = 9-2\sqrt{5}$
- $\sqrt{5+2\sqrt{6}} = \sqrt{3}+\sqrt{2}$
- $\sqrt{5-2\sqrt{6}} = \sqrt{3}-\sqrt{2}$
- $\sqrt{15+2\sqrt{30}+2\sqrt{20}+2\sqrt{6}} = \sqrt{10}+\sqrt{3}+\sqrt{2}$
- $\sqrt{15+2\sqrt{30}-2\sqrt{20}-2\sqrt{6}} = \sqrt{10}+\sqrt{3}-\sqrt{2}$
- $\sqrt{15-2\sqrt{30}-2\sqrt{20}+2\sqrt{6}} = \sqrt{10}-\sqrt{3}-\sqrt{2}$

□ Viète's expansion for π

as an infinite product of nested square roots

$$\frac{2}{\pi} = a_1 a_2 a_3 \cdots$$

wh

$$a_1 = \sqrt{\frac{1}{2}}$$

$$a_2 = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}$$

$$a_3 = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}$$

etc

to pass from a_n to a_{n+1} wh $n \in$ pos int

replace the last $\frac{1}{2}$ by $\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}$

□ the golden ratio

$$\varphi = \frac{1}{2}\left(1 + \sqrt{5}\right)$$

is characterized as

the positive root of the quadratic equation

$$\varphi^2 = \varphi + 1$$

which says remarkably:

to square that number, just add one

- writing the equation $\varphi^2 = \varphi + 1$ in the form

$$\varphi = \sqrt{1+\varphi}$$

and repeatedly substituting $\sqrt{1+\varphi}$ for φ on the RHS

leads to the sequence

$$\varphi = \sqrt{1+\varphi}$$

$$\varphi = \sqrt{1+\sqrt{1+\varphi}}$$

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\varphi}}}$$

etc

& the representation of φ as

the simplest infinite nested square root

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$$

- writing the equation $\varphi^2 = \varphi + 1$ in the form

$$\varphi = 1 + \frac{1}{\varphi}$$

and repeatedly substituting $1 + \frac{1}{\varphi}$ for φ on the RHS

leads to the sequence

$$\varphi = 1 + \frac{1}{\varphi}$$

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{\varphi}}$$

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\varphi}}}$$

etc

& the representation of φ as

the simplest infinite continued fraction

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

□ two equivalent
general infinite nested square root evaluations
& special cases

- $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}} = \frac{1}{2}\left(1+\sqrt{4a+1}\right)$

wh a $\in$ pos real nr

- $\sqrt{a(a-1)+\sqrt{a(a-1)+\sqrt{a(a-1)+\cdots}}} = a$

wh a $\in$ real nr > 1

- $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} = \varphi$

- $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} = 2$

- $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}} = 3$

- $\sqrt{12+\sqrt{12+\sqrt{12+\sqrt{12+\cdots}}}} = 4$

- $\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\cdots}}}} = 5$

□ a general infinite nested square root evaluation
& special cases

- $\sqrt{1 + a\sqrt{1 + (a+1)\sqrt{1 + (a+2)\sqrt{1 + \cdots}}}} \; = \; a + 1$

wh a $\in$ nonneg real nr

- $\sqrt{1 + \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}} \; = \; 2$ for a $= 1$

- $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}} \; = \; 3$ for a $= 2$

- $\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \cdots}}}} \; = \; 4$ for a $= 3$

- $\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + \cdots}}}} \; = \; 5$ for a $= 4$

etc

- note that each numerical line above

when squared & simplified
gives the next line

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☐ a general infinite nested square root evaluation
& special cases

- $\sqrt{a+b\sqrt{a+b\sqrt{a+b\sqrt{a+\cdots}}}} = \frac{1}{2}\left(b+\sqrt{4a+b^2}\right)$

wh a, b $\in$ nonneg real nr

- $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} = \varphi$ for a = 1 & b = 1

- $\sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{1+\cdots}}}} = 1+\sqrt{2}$ for a = 1 & b = 2

- $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} = 2$ for a = 2 & b = 1

- $\sqrt{2+2\sqrt{2+2\sqrt{2+2\sqrt{2+\cdots}}}} = 1+\sqrt{3}$ for a = 2 & b = 2

□ a general infinite nested square root evaluation
& special cases

- $\sqrt{a+\sqrt{b+\sqrt{a+\sqrt{b+\cdots}}}} = x$

$\Leftrightarrow \left(x^2 - a\right)^2 - x - b = 0 \ \& \ x > \sqrt{a}$

wh $a,\ b,\ x \in$ pos real nr

- $\sqrt{1+\sqrt{7+\sqrt{1+\sqrt{7+\cdots}}}} = 2$ for $a = 1,\ b = 7,\ x = 2$

- $\sqrt{1+\sqrt{61+\sqrt{1+\sqrt{61+\cdots}}}} = 3$ for $a = 1,\ b = 61,\ x = 3$

- $\sqrt{2+\sqrt{46+\sqrt{2+\sqrt{46+\cdots}}}} = 3$ for $a = 2,\ b = 46,\ x = 3$

- $\sqrt{3+\sqrt{33+\sqrt{3+\sqrt{33+\cdots}}}} = 3$ for $a = 3,\ b = 33,\ x = 3$

- $\sqrt{4+\sqrt{22+\sqrt{4+\sqrt{22+\cdots}}}} = 3$ for $a = 4,\ b = 22,\ x = 3$

- $\sqrt{5+\sqrt{13+\sqrt{5+\sqrt{13+\cdots}}}} = 3$ for $a = 5,\ b = 13,\ x = 3$

- $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}} = 3$ for $a = 6,\ b = 6,\ x = 3$

- $\sqrt{7+\sqrt{1+\sqrt{7+\sqrt{1+\cdots}}}} = 3$ for $a = 7,\ b = 1,\ x = 3$

- $\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\cdots}}}} = 1.7579$

more here

□ the sine & cosine of the angle $\frac{\pi}{18}$

are expressible ito

infinite nested square roots

- $\sin\frac{\pi}{18} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\cdots}}}} = 0.17365$

wh the pattern of signs is

$-++-++\cdots$

with repeating period $-++$

- $\cos\frac{\pi}{18} = \frac{1}{6}\sqrt{3}\left(1+\sqrt{8-\sqrt{8-\sqrt{8+\sqrt{8-\cdots}}}}\right) = 0.98481$

wh the pattern of signs is

$--+--+\cdots$

with repeating period $--+$

☐ one of Ramanujan's simpler formulas evaluating a general infinite nested square root is derived as follows:

let

$a,\ n,\ x\ \in$ nonneg real nr var

note the identity

$$(x + n + a)^2 \ = \ ax + (n + a)^2 + x(x + 2n + a)$$

taking the square root of each side
and then repeatedly replacing x by x + n
gives

$$x + n + a = \sqrt{ax + (n + a)^2 + x(x + 2n + a)}$$

$$x + 2n + a = \sqrt{a(x + n) + (n + a)^2 + (x + n)(x + 3n + a)}$$

$$x + 3n + a = \sqrt{a(x + 2n) + (n + a)^2 + (x + 2n)(x + 4n + a)}$$

$$x + 4n + a = \sqrt{a(x + 3n) + (n + a)^2 + (x + 3n)(x + 5n + a)}$$

etc

substituing backward
gives the desired expression for
$x + n + a$
as an infinite nested square root

in order to help manage

the unwieldly expressions that arise

define

$$r_0 = ax + (n+a)^2$$

$$r_1 = a(x+n) + (n+a)^2$$

$$r_2 = a(x+2n) + (n+a)^2$$

$$r_3 = a(x+3n) + (n+a)^2$$

etc

$$s_0 = \sqrt{r_0}$$

$$s_1 = \sqrt{r_0 + x\sqrt{r_1}}$$

$$s_2 = \sqrt{r_0 + x\sqrt{r_1 + (x+n)\sqrt{r_2}}}$$

$$s_3 = \sqrt{r_0 + x\sqrt{r_1 + (x+n)\sqrt{r_2 + (x+2n)\sqrt{r_3}}}}$$

etc

then

$$x + n + a = \sqrt{r_0 + x\sqrt{r_1 + (x+n)\sqrt{r_2 + (x+2n)\sqrt{r_3 + \cdots}}}}$$

which is defined to be $\lim_{k \to \infty} s_k$

wh k $\in$ nonneg int var

□ some special cases of the above Ramanujan formula

taking a = 0 & n = 1

- $x + 1 = \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)\sqrt{1 + (x+3)\sqrt{1 + \cdots}}}}}$

wh x ∈ nonneg real nr

taking x = 2

- $3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \cdots}}}}}$

D. constructibility

- a geometric figure in the euclidean plane

such as a polygon or an angle

is said to be

constructible by

unmarked straightedge & adjustable compass

= constructible by straightedge & compass

= constructible by ruler and compass

= constructible by Platonic instruments

= Platonically constructible

= constructible

iff

the figure can be constructed
by finitely many applications of these two instruments
viz
using the unmarked straightedge
to draw the straight line
passing thru two given distinct points
&
using the adjustable compasses
to draw the circle
with a given point as center
and passing thru a given point

☐ T. constructible angles

let

- $\alpha \in$ angle in the euclidean plane

then

tfsape

- α is Platonically constructible

- the six basic trig fcns of α are each quadratically constructible

- some one basic trig fcn of α is quadratically constructible

☐ the sine & cosine of some constructible angles are given below;
these constructible angles are

15^o, 18^o, 30^o,36^o, 45^o, 54^o, 60^o, 72^o, 75^o

$\frac{\pi^r}{2^n}$ wh n $\in$ pos int

• drawing the diagonal of a square
which is 1 unit on a side
gives

$\sin 45^o = \cos 45^o$

$= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

• drawing the altitude of an equilateral triangle
which is 2 units on a side
gives

$\sin 60^o = \cos 30^o = \frac{\sqrt{3}}{2}$

$\cos 60^o = \sin 30^o = \frac{1}{2}$

- by repeated use of the trig identities

$$\sin\frac{\vartheta}{2} = \sqrt{\frac{1-\cos\vartheta}{2}} \quad \& \quad \cos\frac{\vartheta}{2} = \sqrt{\frac{1+\cos\vartheta}{2}} \quad \left(\frac{\vartheta}{2} \in \mathrm{QI}\right)$$

it follows that

$$\sin\frac{\pi}{4} = \frac{1}{2}\sqrt{2}$$

$$\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

$$\sin\frac{\pi}{16} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$$

$$\sin\frac{\pi}{32} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

etc

$$\cos\frac{\pi}{4} = \frac{1}{2}\sqrt{2}$$

$$\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

$$\cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\cos\frac{\pi}{32} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

etc

- $\sin 75^o = \cos 15^o$

$$= \cos\left(45^o - 30^o\right)$$

$$= \cos 45^o \cos 30^o + \sin 45^o \sin 30^o$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$$

$$= \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$$

$$= \frac{1}{4}\sqrt{2}\left(\sqrt{3} + 1\right)$$

- $\sin 75^o = \cos 15^o$

$$= \sqrt{\frac{1 + \cos 30^o}{2}}$$

$$= \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$

$$= \frac{1}{2}\sqrt{2 + \sqrt{3}}$$

- note $\sqrt{6} + \sqrt{2} = \sqrt{2}\left(\sqrt{3} + 1\right) = 2\sqrt{2 + \sqrt{3}}$

- $\cos 75^{o} = \sin 15^{o}$

$$= \sin\left(45^{o} - 30^{o}\right)$$

$$= \sin 45^{o} \cos 30^{o} - \cos 45^{o} \sin 30^{o}$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}$$

$$= \frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)$$

$$= \frac{1}{4}\sqrt{2}\left(\sqrt{3} - 1\right)$$

- $\cos 75^{o} = \sin 15^{o}$

$$= \sqrt{\frac{1 - \cos 30^{o}}{2}}$$

$$= \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}}$$

$$= \frac{1}{2}\sqrt{2 - \sqrt{3}}$$

- note $\sqrt{6} - \sqrt{2} = \sqrt{2}\left(\sqrt{3} - 1\right) = 2\sqrt{2 - \sqrt{3}}$

- consequently

$$\sin 75^{o} \;=\; \cos 15^{o}$$

$$= \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

$$= \frac{1}{4}\sqrt{2}\left(\sqrt{3}+1\right)$$

$$\cos 75^{o} \;=\; \sin 15^{o}$$

$$= \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$$

$$= \frac{1}{4}\sqrt{2}\left(\sqrt{3}-1\right)$$

- how to compute $\cos 36^{o}$ exactly

using only trig & algebra

set $A = 36^{o}$

then

$$5A = 180^{o}$$

$$3A = 180^{o} - 2A$$

$$\cos 3A = -\cos 2A$$

$$\cos 3A + \cos 2A = 0$$

$$4\cos^3 A - 3\cos A + 2\cos^2 A - 1 = 0$$

$$4\cos^3 A + 2\cos^2 A - 3\cos A - 1 = 0$$

set $x = \cos A$

then

$$4x^3 + 2x^2 - 3x - 1 = 0$$

$$(x+1)(4x^2 - 2x - 1) = 0$$

$$4x^2 - 2x - 1 = 0$$

$$x = \frac{2+\sqrt{20}}{8} = \frac{1+\sqrt{5}}{4} = \frac{\varphi}{2}$$

$$\therefore \cos 36^{o} = \frac{1+\sqrt{5}}{4} = \frac{\varphi}{2}$$

• how to compute $\cos 36^\circ$ exactly
using a little bit of geometry

consider an isosceles triangle
with apex angle $= 36^\circ$
with each base angle $= 72^\circ$;
bisect a base angle
& consider how the bisector divides the opposite side;
take the segment with endpoint at the vertex to be x
& the segment with endpoint at the base to be 1;
by similar triangles

$\frac{x+1}{x} = \frac{x}{1}$ which is the golden ratio proportion & thus

$x = \varphi$;

by the law of sines

$\frac{\sin 36^\circ}{1} = \frac{\sin 72^\circ}{\varphi} = \frac{2 \sin 36^\circ \cos 36^\circ}{\varphi}$ & thus

$\cos 36^\circ = \frac{\varphi}{2} = \frac{1+\sqrt{5}}{4}$

- consequently

$$\sin 54^{\circ} = \cos 36^{\circ}$$

$$= \frac{1}{4}\left(1+\sqrt{5}\right)$$

$$= \frac{\varphi}{2}$$

$$\cos 54^{\circ} = \sin 36^{\circ}$$

$$= \frac{1}{4}\sqrt{10-2\sqrt{5}}$$

$$= \frac{1}{2}\sqrt{3-\varphi}$$

- $\sin 72^{\circ} = \cos 18^{\circ}$

$$= \sqrt{\frac{1+\cos 36^{\circ}}{2}}$$

$$= \sqrt{\frac{1+\frac{1+\sqrt{5}}{4}}{2}}$$

$$= \frac{1}{4}\sqrt{10+2\sqrt{5}}$$

$$= \frac{1}{2}\sqrt{\varphi+2}$$

- $\cos 72^{\circ} = \sin 18^{\circ}$

$$= \sqrt{\frac{1-\cos 36^{\circ}}{2}}$$

$$= \sqrt{\frac{1-\frac{1+\sqrt{5}}{4}}{2}}$$

$$= \frac{1}{4}\sqrt{6-2\sqrt{5}}$$

$$= \frac{1}{4}\left(\sqrt{5}-1\right)$$

$$= \frac{1}{2}\left(\varphi-1\right)$$

- consequently

$$\sin 72^{o} = \cos 18^{o}$$

$$= \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$$

$$= \frac{1}{2}\sqrt{\varphi + 2}$$

$$\cos 72^{o} = \sin 18^{o}$$

$$= \frac{1}{4}\left(\sqrt{5} - 1\right)$$

$$= \frac{1}{2}(\varphi - 1)$$

T. Gauss' s regular polygon constructibility theorem

let

- n $\in$ int $\geq$ 3
- $\alpha = \dfrac{2\pi^{r}}{n} = \dfrac{360^{o}}{n}$

then

tfsape

- the regular polygon of n sides is

Platonically constructible

- the angle α is

Platonically constructible

- some basic trig fcn of α is

quadratically constructible

- all six basic trig fcns of α are

quadratically constructible

- n = a product of

a nonnegative integer power of 2

& distinct Fermat primes

D. Fermat numbers

let

- n $\in$ nonneg int

then

- the Fermat number of index n

= the nth Fermat number

$=_{dn}$ F_n wh F ← Fermat

$=_{df}$ $2^{2^n} + 1$

R. the only known Fermat primes as of the year 2002 are the first five Fermat numbers

viz

$F_0 = 2^1 + 1 = 3$

$F_1 = 2^2 + 1 = 5$

$F_2 = 2^4 + 1 = 17$

$F_3 = 2^8 + 1 = 257$

$F_4 = 2^{16} + 1 = 65537$

□ a whiff of nested radicals involving cube roots

problem: to evaluate $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$

solution: the golden ratio

$$\varphi \;=\; \frac{1}{2}\left(1+\sqrt{5}\right)$$

has the property that

$$\varphi^2 \;=\; \varphi+1;$$

multiplying and dividing

repeatedly by φ

and simplifying

gives

etc

$$\varphi^5 = 5\varphi + 3$$

$$\varphi^4 = 3\varphi + 2$$

$$\varphi^3 = 2\varphi + 1$$

$$\varphi^2 = \varphi + 1$$

$$\varphi = \varphi$$

$$\frac{1}{\varphi} = \varphi - 1$$

$$\frac{1}{\varphi^2} = 2 - \varphi$$

$$\frac{1}{\varphi^3} = 2\varphi - 3$$

$$\frac{1}{\varphi^4} = 5 - 3\varphi$$

$$\frac{1}{\varphi^5} = 5\varphi - 8$$

etc

note that the coefficients of the first degree polynomials in φ
are members of the Fibonacci sequence,
alternating in sign for the powers of the reciprocal of φ

hence

$$2+\sqrt{5} \;=\; 2\varphi+1 \;=\; \varphi^3$$

$$2-\sqrt{5} \;=\; 3-2\varphi \;=\; -\frac{1}{\varphi^3}$$

$$\sqrt[3]{2+\sqrt{5}} \;=\; \varphi$$

$$\sqrt[3]{2-\sqrt{5}} \;=\; -\frac{1}{\varphi} \;=\; 1-\varphi$$

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}} \;=\; 1$$

☐ multiplicative nested pth powers

- the basic iterated step:

replace something by something times a pth power

- start with

$a_0 = x_0$

- replace x_0 by $x_0 \times x_1^p$ to get

$a_1 = x_0 \times x_1^p$

- replace x_1 by $x_1 \times x_2^p$ to get

$a_2 = x_0 \times \left(x_1 \times x_2^p\right)^p$

- replace x_2 by $x_2 \times x_3^p$ to get

$a_3 = x_0 \times \left(x_1 \times \left(x_2 \times x_3^p\right)^p\right)^p$

etc

• if the process is ended in fnitely many steps,
then a finite multiplicative nested pth power
is obtained;
if the process continues to
an infinite sequence of the a' s,
then an infinite muliplicative nested pth power
is obtained
and the limit of the a' s if it exists
viz
$\lim_{k \to \infty} a_k$ (wh k $\in$ nonneg int var) iie
is denoted

$$x_0 \times \left(x_1 \times \left(x_2 \times (x_3 \times \cdots)^p \right)^p \right)^p$$

☐ for infinite multiplicative nested pth powers
as above

- if $p = 1$, then the a's are

the partial products of an infinite product

$$x_0 \times x_1 \times x_2 \times x_3 \times \cdots$$

$$= x_0 x_1 x_2 x_3 \cdots$$

$$= \prod_{n=0}^{\infty} x_n$$

viz

$a_0 = x_0$

$a_1 = x_0 x_1$

$a_2 = x_0 x_1 x_2$

$a_3 = x_0 x_1 x_2 x_3$

etc

☐ here are some particular examples of infinite multiplicative nested radicals

for any pos real nr x

- $x = \sqrt{x\sqrt{x\sqrt{x\sqrt{x\cdots}}}}$

- $x = \sqrt[3]{x^2\sqrt[3]{x^2\sqrt[3]{x^2\sqrt[3]{x^2\cdots}}}}$

etc
&
ing

- $x = \sqrt[n]{x^{n-1}\sqrt[n]{x^{n-1}\sqrt[n]{x^{n-1}\sqrt[n]{x^{n-1}\cdots}}}}$

wh $n \in$ plural int