Real Second-degree Binary Polynomial Equations

#79 of Gottschalk's Gestalts

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GG79-2

□ the general real second - degree polynomial equation
in the two real number variables x and y
in canonical form
$=_{df}$
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (*)$$
wh
$A, B, C, D, E, F \in$ real nr
st
$$A \neq 0 \lor B \neq 0 \lor C \neq 0$$

□ if the real number variables x and y
are interpreted as coordinates,
abscissa and ordinate,
of a point P in the euclidean plane provided with
a rectangular coordinate system with x - axis and y - axis,
then
the graph of (*)
= the set of all points $P(x, y)$ satisfying (*)
is a conic section
or two parallel straight lines
or the empty set;
the graph classification theorem below
spells out this fact in detail
according to the nature of the coefficients of (*)

□ a conic section

$=_{df}$ a plane section

of a two - nappe right circular cone

in euclidean 3 - space

☐ a conic section
is one of the following curves:

• an ellipse = a proper ellipse
which has
two distinct foci & two distinct directrices
& unequal positive semiaxes
& positive eccentricity < 1
& a center of symmetry
& two axes of symmetry
and
which is
a convex simple closed curve

• a hyperbola = a proper hyperbola
which has
two distinct foci & two distinct directrices
& positive semiaxes
& eccentricity > 1
& a pair of intersecting straight - line asymptotes
& a center of symmetry
& two axes of symmetry
& two branches
which are convex simple open
unbounded congruent arcs

• a parabola = a proper parabola
which has
one focus & one directrix with focus $\notin$ directrix
& positive latus rectum
& eccentricity = 1
& no center of symmetry
& one axis of symmetry
and
which is
a convex simple open unbounded curve,

• a circle = a proper circle
which has
a positive radius
(a circle may be considered to be
a degenerate / limiting form
of an ellipse
with coincident foci
& equal semiaxes
& eccentricity = 0)

• a point
(a point may be considered to be
a degenerate / limting form
of an ellipse with zero semiaxes
or
of a circle with zero radius)

• one straight line
(may be considered to be
a degenerate / limiting form
of a parabola
with zero curvature)

• two intersecting straight lines
(may be considered to be
a degenerate / limiting form
of a hyperbola
which coincides with its asymptotes)

• conversely,
any such curve is congruent to
a plane section of a right circular cone

□ strictly speaking
two parallel straight lines and the empty set
are not conic sections;
but they are plane sections
of the degenerate / limiting form
of a right circular cone
when the vertex recedes to infinity
viz
a right circular cylinder

• a plane section of a right circular cylinder
is one of the following curves:
empty set,
a single line,
two parallel lines,
a circle,
an ellipse;
conversely,
any such curve is congruent to
a plane section of a right circular cylinder

• a single straight line
and
a pair of parallel straight lines
may each be considered to be
a degenerate / limiting form
of an ellipse
when the foci go to infinity
in the opposite directions

□ the following functions
d, q, t, s
of the coefficients of (*)
are called
invariants of (*)
because their values remain unchanged
when the coordinate axes
are translated and rotated
to obtain a new equation;
when (*) is multiplied by a nonzero real number k,
the new equation has the same graph
as the original equation
but the original
d is multiplied by k^3,
q is multiplied by k^2,
t is multiplied by k,
s remains unchanged

• the discriminant of (*)

$=_{dn}$ d

$$=_{df} \begin{vmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{vmatrix}$$

• the quadratic discriminant of (*)

$=_{dn}$ q

$$=_{df} \begin{vmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{vmatrix}$$

• the quadratic trace of (*)

$=_{dn}$ t

$=_{df}$ $A + C$

- the sign invariant of (*)

$=_{dn}$ s

$$=_{df} \operatorname{sgn}\left(\begin{vmatrix} A & \frac{D}{2} \\ \frac{D}{2} & F \end{vmatrix} + \begin{vmatrix} C & \frac{E}{2} \\ \frac{E}{2} & F \end{vmatrix} \right)$$

- note that
the three second - order determinants
appearing in the definitions of q & s
are the minors of the diagonal entries of
the third - order determinant
appearing in the definition of d

- $t^2 = 4q \Leftrightarrow A = C \;\&\; B = 0$

□ The Graph Classification Theorem
for Real Second - degree Polynomial Equations
in Two Variables

the graphs of the equations in the form (*)
are described
in the following six exclusive & exhaustive cases:

Δ case I. $d \neq 0$ & $q > 0$

• subcase I_1. $\operatorname{sgn} d \neq \operatorname{sgn} t$ & $4q \neq t^2$

$\Rightarrow$

graph = an ellipse

• subcase I_2. $\operatorname{sgn} d \neq \operatorname{sgn} t$ & $4q = t^2$

$\Rightarrow$

graph = a circle

• subcase I_3. $\operatorname{sgn} d = \operatorname{sgn} t$

$\Rightarrow$

graph = $\varnothing$

Δ case II. $d \neq 0$ & $q < 0$

$\Rightarrow$

graph = a hyperbola

Δ case III. $d \neq 0$ & $q = 0$

$\Rightarrow$

graph = a parabola

Δ case IV. $d = 0$ & $q > 0$

$\Rightarrow$

graph = one point

Δ case V. $d = 0$ & $q < 0$

$\Rightarrow$

graph = the union of

two distinct intersecting straight lines

Δ case VI. $d = 0$ & $q = 0$

• subcase VI_1. $s > 0$

$\Rightarrow$

graph $= \varnothing$

• subcase VI_2. $s < 0$

$\Rightarrow$

graph = the union of
two distinct parallel straight lines

• subcase VI_3. $s = 0$

$\Rightarrow$

graph = one straight line