Identities in Commutative Rings

#75 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2003

GG75-1 (32)

500 Angell St #414
Providence RI 02906

GG75-2

identities in commutative rings

☐ this is a short semi - systematic listing
of various necessarily algebraic identities
in commutative rings

☐ standing notation
- R ∈ com ring   wh R ← r̲ing
- a, b, c, d, p, q, r, s, x, y, z (perhaps adfixed) ∈ var R
- n ∈ pos int

☐ here an identity is understood to be
an equality of two ring expressions
that is true for all values of the variables

☐ identities may be roughly classified as
- factorization:

to convert an algebraic sum into a product;
- expansion:

to convert a product into an algebraic sum;
- change - of - form

□ factoring = factorizing
&
expanding
are opposite / inverse procedures;
given an equation that is read from left to right
for one procedure,
then reading the equation from right to left
is the other procedure;
the results are called
factorizations
&
expansions

□ the simplest example of
factoring / factorization & expanding / expansion
is to be found in the distributive axiom / law of rings
$a(b + c) = ab + ac$
which connects
the additive & multiplicative structures of a ring;
factoring / factorization: from RHS to LHS;
expanding / expansion: from LHS to RHS

☐ factoring a difference of like odd powers

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

$$a^5 - b^5 = (a-b)\left(a^4 + a^3b + a^2b^2 + ab^3 + b^4\right)$$

$$a^7 - b^7 = (a-b)\left(a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6\right)$$

etc

$$a^{2n+1} - b^{2n+1} = (a-b)\sum_{i=0}^{2n} a^{2n-i}b^i$$

☐ factoring a sum of two like odd powers

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$

$$a^5 + b^5 = (a+b)\left(a^4 - a^3b + a^2b^2 - ab^3 + b^4\right)$$

$$a^7 + b^7 = (a+b)\left(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6\right)$$

etc

$$a^{2n+1} + b^{2n+1} = (a+b)\sum_{i=0}^{2n}(-1)^i a^{2n-i} b^i$$

☐ factoring a difference of like even powers

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^4 - b^4 = (a+b)(a-b)\left(a^2 + b^2\right)$$

$$a^6 - b^6 = (a+b)(a-b)\left(a^2 + ab + b^2\right)\left(a^2 - ab + b^2\right)$$

$$a^8 - b^8 = (a+b)(a-b)\left(a^2 + b^2\right)\left(a^4 + b^4\right)$$

$$a^{10} - b^{10} = (a+b)(a-b)$$
$$\times\left(a^4 + a^3b + a^2b^2 + ab^3 + b^4\right)\left(a^4 - a^3b + a^2b^2 - ab^3 + b^4\right)$$

etc

the pattern is determined by the exponent
expressed as a power of 2 times an odd integer

□ factoring a sum of two squares
with a special element

$i \in R$ & $i^2 = -1$

$\Rightarrow$

$a^2 + b^2 = (a + ib)(a - ib)$

□ factoring a sum of two fourth powers
with special elements

• $i \in R$ & $i^2 = -1$

$\Rightarrow$

$$a^4 + b^4 = \left(a^2 + ib^2\right)\left(a^2 - ib^2\right)$$

• $i, j \in R$ & $i^2 = -1$ & $j^2 = -i$

$\Rightarrow$

$$a^4 + b^4 = (a + jb)(a - jb)\left(a^2 - ib^2\right)$$

• $i, k \in R$ & $i^2 = -1$ & $k^2 = i$

$\Rightarrow$

$$a^4 + b^4 = \left(a^2 + ib^2\right)(a + kb)(a - kb)$$

• $i, j, k \in R$ & $i^2 = -1$ & $j^2 = -i$ & $k^2 = i$

$\Rightarrow$

$$a^4 + b^4 = (a + jb)(a - jb)(a + kb)(a - kb)$$

□ semisum factorizations

• for a semisum of two elements

$$2s = a + b$$

$$\Rightarrow$$

$$4a^2b^2 - (a^2 + b^2)^2$$

$$= 16s^2(s - a)(s - b)$$

• for a semisum of three elements

$$2s = a + b + c$$

$$\Rightarrow$$

$$4a^2b^2 - (a^2 + b^2 - c^2)^2$$

$$= 16s(s - a)(s - b)(s - c)$$

• for a semisum of four elements

$$2s = a + b + c + d$$

$$\Rightarrow$$

$$4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$$

$$= 16(s - a)(s - b)(s - c)(s - d)$$

□ some special factorings

$$a^4 + a^2b^2 + b^4$$
$$= \left(a^2 + ab + b^2\right)\left(a^2 - ab + b^2\right)$$

$$a^8 + a^4b^4 + b^8$$
$$= \left(a^4 + a^2b^2 + b^4\right)\left(a^4 - a^2b^2 + b^4\right)$$
$$= \left(a^2 + ab + b^2\right)\left(a^2 - ab + b^2\right)\left(a^4 - a^2b^2 + b^4\right)$$

$$a^{12} + a^6b^6 + b^{12}$$
$$= \left(a^6 + a^3b^3 + b^6\right)\left(a^6 - a^3b^3 + b^6\right)$$

etc

$$a^{4n} + a^{2n}b^{2n} + b^{4n}$$
$$= \left(a^{2n} + a^nb^n + b^{2n}\right)\left(a^{2n} - a^nb^n + b^{2n}\right)$$

☐ binomial expansion / formula / theorem

$$(a+b)^2 = a^2 + ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

etc

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

□ squares of multinomials

$$(a+b)^2$$
$$= a^2 + b^2 + 2ab$$

$$(a+b+c)^2$$
$$= a^2 + b^2 + c^2 + 2(ab + ac + bc)$$

$$(a+b+c+d)^2$$
$$= a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$$

etc

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i{}^2 + 2\sum_{1\le i<j\le n} a_i a_j$$

there is a multinomial expansion / formula / theorem
generalizing the binomial expansion / formula / theorem
but it is rather complicated to write down

□ some special expansions

$$(a+b)^2+(b+c)^2+(c+a)^2$$
$$=2(a^2+b^2+c^2+ab+bc+ca)$$

$$(a-b)^2+(b-c)^2+(c-a)^2$$
$$=2(a^2+b^2+c^2-ab-bc-ca)$$

□ identities involving squares
of algebraic sums of squares

$$\left(a^2+b^2\right)^2=\left(a^2-b^2\right)^2+(2ab)^2$$

$$\left(a^2+b^2+c^2+d^2\right)^2$$

$$=$$

$$\left(a^2+b^2-c^2-d^2\right)^2+(2ac+2bd)^2+(2ad-2bc)^2$$

☐ the quadratic formula in disguise

$$p^2 = b^2 - 4ac$$

$$\Rightarrow$$

$$4a(ax^2 + bx + c)$$

$$= (2ax + b + p)(2ax + b - p)$$

□ the Fibonacci two - square identity:
the product of two sums of two squares
is a sum of two squares

$$\left(a^2 + b^2\right)\left(c^2 + d^2\right) = (ac - bd)^2 + (ad + bc)^2$$

this identity in the real field
is a virtual restatement of the fact that
the absolute value of the product
of two complex numbers
equals
the product of the absolute values
of the complex numbers

☐ the Euler four - square identity:
the product of two sums of four squares
is a sum of four squares

$$\left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)\left(b_1^2 + b_2^2 + b_3^2 + b_4^2\right)$$
$$=$$
$$\left(a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4\right)^2$$
$$+$$
$$\left(a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3\right)^2$$
$$+$$
$$\left(a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2\right)^2$$
$$+$$
$$\left(a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1\right)^2$$

this identity in the real field
is a virtual restatement of the fact that
the norm of the product of two quaternions
equals
the product of the norms of the quaternions

☐ the Degen eight - square identity:
the product of two sums of eight squares
is a sum of eight squares

in compresssed notation:
for four quaternions $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$

$$\left(|\mathbf{a}|^2 + |\mathbf{b}|^2\right)\left(|\mathbf{c}|^2 + |\mathbf{d}|^2\right) = |\mathbf{ac} - \mathbf{d}\overline{\mathbf{b}}|^2 + |\overline{\mathbf{a}}\mathbf{d} + \mathbf{cb}|^2$$

this identity in the real field
is a virtual restatement of the fact that
the norm of the product of two octonions
equals
the product of the norms of the octonions

note: the product of the two octonions
in the compressed notation is

$$(\mathbf{a}, \mathbf{b})(\mathbf{c}, \mathbf{d}) = (\mathbf{ac} - \mathbf{d}\overline{\mathbf{b}},\ \overline{\mathbf{a}}\mathbf{d} + \mathbf{cb})$$

wh an octonion is considered to be
an ordered pair of quaternions

□ the Ferrari identity

$$\left(a^2 - b^2 - 2bc + 2ca\right)^4$$

$$+$$

$$\left(b^2 - c^2 - 2ca - 2ab\right)^4$$

$$+$$

$$\left(c^2 - a^2 + 2ab + 2bc\right)^4$$

$$=$$

$$2\left(a^2 + b^2 + c^2 - ab + bc + ca\right)^4$$

□ an Euler identity

$$\left(abp^2 + cdq^2\right)\left(acr^2 + bds^2\right)$$

$$=$$

$$ad(bps \pm cqr)^2 + bc(apr \mp dqs)^2$$

this identity generalizes
the Fibonacci identity

□ the trinomial identity

$$x = pr - bqs$$

&

$$y = qr + ps + aqs$$

$\Rightarrow$

$$\left(p^2 + apq + bq^2\right)\left(r^2 + ars + bs^2\right) = x^2 + axy + by^2$$

☐ identity involving cubes of binomials

$$a(a+2b)^3 = a(a-b)^3 + b(a-b)^3 + b(2a+b)^3$$

☐ the Liouville polynomial identity

$$6\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^2$$

$$=$$

$$(x_1 + x_2)^4 + (x_1 + x_3)^4 + (x_1 + x_4)^4$$

$$+ (x_2 + x_3)^4 + (x_2 + x_4)^4 + (x_3 + x_4)^4$$

$$+ (x_1 - x_2)^4 + (x_1 - x_3)^4 + (x_1 - x_4)^4$$

$$+ (x_2 - x_3)^4 + (x_2 - x_4)^4 + (x_3 - x_4)^4$$

□ two Ramanujan identities

$$\left(a^2 + 7ab - 9b^2\right)^3 + \left(2a^2 - 4ab + 12b^2\right)^3$$
$$=$$
$$\left(2a^2 + 10b^2\right)^3 + \left(a^2 - 9ab - b^2\right)^3$$

$$\left(4a^5 - 5a\right)^4 + \left(6a^4 - 3\right)^4 + \left(4a^4 + 1\right)^4$$
$$=$$
$$\left(4a^5 + a\right)^4 + \left(2a^4 - 1\right)^4 + 3^4$$

□ the Lagrange identity

$$\left(\sum_{i=1}^{n} a_i^{2}\right)\left(\sum_{i=1}^{n} b_i^{2}\right)-\left(\sum_{i=1}^{n} a_i b_i\right)^2$$

$$=$$

$$\sum_{1\le i<j\le n} \left(a_i b_j - a_j b_i\right)^2$$

☐ the Binet - Cauchy identity

$$\left(\sum_{i=1}^{n} a_i c_i\right)\left(\sum_{i=1}^{n} b_i d_i\right) - \left(\sum_{i=1}^{n} a_i d_i\right)\left(\sum_{i=1}^{n} b_i c_i\right)$$

$$= \sum_{1 \le i < j \le n} \left(a_i b_j - a_j b_i\right)\left(c_i d_j - c_j d_i\right)$$

this identity generalizes
the Lagrange identity

□ the notion of nth order determinant
& some of the standard properties
of nth order determinants
carry over to com rings
eg
for 2nd order determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} =_{df} ad - bc$$

&

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} = \begin{vmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{vmatrix}$$

□ the notion of n - vector over R
is definable as
an ordered n - tuple of elements of R;
vectors may be added or subtracted or
multiplied by ring elements
on the left or on the right
componentwise;
the dot product is defined as usual and
the cross product of two 3 - vectors is defined as usual;
many of the algebraic vector identities
for the reals say
now also ensue for R;
eg Lagrange's identity for four 3 - vectors

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

expressed suggestively as
dot of crosses
equals
determinant of dots

□ the elementary symmetric functions / polynomials

= the sigmas

- for 2 variables $r_1,\ r_2$

$(x-r_1)(x-r_2)$

$=x^2-(r_1+r_2)x+r_1r_2$

$=x^2-\sigma_1x+\sigma_2$

wh

$\sigma_1=r_1+r_2$

$\sigma_2=r_1r_2$

- for 3 variables $r_1,\ r_2,r_3$

$(x-r_1)(x-r_2)(x-r_3)$

$=x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_2r_3+r_3r_1)x-r_1r_2r_3$

$=x^3-\sigma_1x^2+\sigma_2x-\sigma_3$

wh

$\sigma_1=r_1+r_2+r_3$

$\sigma_2=r_1r_2+r_2r_3+r_3r_1$

$\sigma_3=r_1r_2r_3$

etc

• the fundamental theorem
on symmetric functions / polynomials
states that
any symmetric polynomial in the r's
equals
a polynomial in the σ's
eg
$r_1^2 + r_2^2 = \sigma_1^2 - 2\sigma_2$ for two variables
$r_1^2 + r_2^2 + r_3^2 = \sigma_1^2 - 2\sigma_2$ for three variables
& likewise for any number of variables;
Newton's identities concern the σ's

□ for a com unital ring R

$$a^2 + b^2 = 1$$

$$\Rightarrow$$

$$\left(a^6 + 1\right)\left(b^2 + 1\right) = \left(a^2 + 1\right)\left(b^6 + 1\right)$$

this implication could be called
a conditional identity

this conditional identity
gives the trig identity

$$\frac{\sin^6 A + 1}{\sin^2 A + 1} = \frac{\cos^6 A + 1}{\cos^2 A + 1}$$