Comments on the Three Big C's of General Topology: Compact, Connected, Continuous

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C. compact (adjective) / compactness (noun)
is
the topological generalization of
finite (adjective) / finiteness (noun);
iow
the notion of compact space
is
the topological generalization of
the notion of finite set;
two precise topological facts
in support of this philosophical insight
are:
a discrete space is compact iff it is finite
&
the definition of compact space
contains a central use of finite sets

D. compact spaces

let

$X \in$ top sp

then

X is a compact topological space

= X is a compact space

= X is compact

$=_{df}$ any one of the following pairwise equivalent statements is satisfied

(1) if $\mathcal{A}$ is an open cluster on X st

$\bigcup \mathcal{A} = X$

then there exists a finite subcluster $\mathcal{B}$ of $\mathcal{A}$ st

$\bigcup \mathcal{B} = X$

(1') if $\mathcal{A}$ is a closed cluster on X st

$\bigcap \mathcal{A} = \varnothing$

then there exists a finite subcluster B of A st

$\bigcap \mathcal{B} = \varnothing$

(2) if $\mathcal{A}$ is an open cluster on X st

$\bigcup \mathcal{B} \neq X$

for every finite subcluster $\mathcal{B}$ of $\mathcal{A}$,

then

$\bigcup \mathcal{A} \neq X$

(2') if $\mathcal{A}$ is a closed cluster on X st

$\bigcap \mathcal{B} \neq \emptyset$

for every finite subcluster $\mathcal{B}$ of $\mathcal{A}$,

then

$\bigcap \mathcal{A} \neq \emptyset$

(3) if $\mathcal{A}$ is an open cluster on X
with the finite union property,
then
$\bigcup \mathcal{A} \neq$

(3') if A is a closed cluster on X
with the finite intersection property,
then
$\bigcap \mathcal{A} \neq \varnothing$

(4) if F is a filter on X,

then

$\bigcap \overline{\mathcal{F}} \neq \varnothing$

(5) if $\mathcal{F}$ is a filterbase on X,

then

$\bigcap \overline{\mathcal{F}} \neq \varnothing$

(6) if F is a filtersubbase on X,

then

$\bigcap \overline{\mathcal{F}} \neq \varnothing$

(7) every ultrafilter on X is convergent

C. verbal paraphrases of statements
in the preceding definition

(1) every open cover contains a finite cover

(1) every open cover is reducible to a finite cover

(1') every closed cluster with empty intersection
contains a finite cluster with empty intersection

(2), (2') are contrapositives of (1), (1')

(3), (3') are terminological variants of (2), (2')

(4), (5), (6). (7) are in the language of filters
and are related to (3')

the duals of (4), (5), (6), (7)
would be in the language of dualfilters
and related to (3)

C. below are several notable theorems
in purely verbal form
about compact spaces

T. every closed subset of a compact space
is compact

T. every compact subset of a Hausdorff space
is closed

T. every continuous image of a compact space
is compact

T. every one-to-one continuous function
on a compact space to a Hausdorff space
is homeomorphic into

T. every finite cartesian sum of compact spaces
is compact

T. every cartesian product of compact spaces
is compact

T. every real-valued continuous function
on a compact space
assumes both
an absolute maximum value
and
an absolute minimum value

T. every continuous function
on a compact uniform space to a uniform space
is uniformly continuous

T. every continuous function
on a uniform space to a uniform space
is uniformly continuous
over every compact subset of the domain space

T. every one-to-one continuous function
on a compact uniform space
to a uniform Hausdorff space
is unimorphic into

C. connected (adjective) / connectedness (noun)

is

the exact topological expression

of the intuitive but vague notion of

' being in one piece'

' hanging together'

D. connected spaces

let
X $\in$ top sp

then
X is a connected topological space
= X is a connected space
= X is connected
$=_{df}$ any one of the following
pairwise equivalent statements
is satisfied

- the only clopen subsets of X are
the space X and the empty set ∅

- there does not exist
a nonempty nonspacial clopen subet of X

- there does not exist
a closed binary partition of X

- there does not exist
an open binary partition of X

- there does not exist
a clopen binary partition of X

• if $\{A,\ B\}$ is a binary partition of X,

then

$$\left(A \cap \overline{B}\right) \cup \left(\overline{A} \cap B\right) \neq \varnothing$$

wiet

$$A \cap \overline{B} \neq \varnothing \text{ or } \overline{A} \cap B \neq \varnothing$$

• if $\{A,\ B\}$ is a binary partition of X,

then

$$\left(A \cup \overset{o}{B}\right) \cap \left(\overset{o}{A} \cup B\right) \neq X$$

wiet

$$A \cup \overset{o}{B} \neq X \text{ or } \overset{o}{A} \cup B \neq X$$

C. below are several notable theorems
in purely verbal form
about connected spaces

T. every continuous image of a connected space
is connected

T. every cartesian product of connected spaces
is connected

T. every real-valued continuous function
on a connected space
assumes all intermediate values

T. characterizations of continuous functions

let

- (X, $\mathcal{T}$), (Y, $\mathcal{S}$) $\in$ top sp
- φ:X $\to$ Y

then

tfsape

- φ is continuous
- φ is continuous on X
- φ is pointwise continuous
- φ is pointwise continuous on X
- φ is continuous at each point of X

• if x is a point of X, and
if V is a neighborhood of $x\varphi$,
then there exists a neighborhood U of x st
$U\varphi \subset V$

• $\forall x \in X . \forall V \in N_{x\varphi} . \exists U \in N_x . U\varphi \subset V$

• if x is a point of X, and
if V is a neighborhood of $x\varphi$,
then $V\varphi^{-1}$ is a neighborhood of x

• $\forall x \in X . \forall V \in \mathcal{N}_{x\varphi} . V\varphi^{-1} \in \mathcal{N}_x$

• $\forall x \in X . \mathcal{N}_{x\varphi}\varphi^{-1} \subset \mathcal{N}_x$

• if x is a point of X,
then there exists a subbase $\mathcal{B}$ of $\mathcal{N}_{x\varphi}$ st
$\mathcal{B}\varphi^{-1} \subset \mathcal{N}_x$

- if E is an open subset of Y, then $E\varphi^{-1}$ is an open subset of X

- if E is a closed subset of Y, then $E\varphi^{-1}$ is a closed subset of X

- $\mathcal{S}\varphi^{-1} \subset \mathcal{T}$

- there exists a subbase $\mathcal{B}$ of $\mathcal{S}$ st $\mathcal{B}\varphi^{-1} \subset \mathcal{T}$

- if x is a point of X,
if A is a subset of X, and
if x is adherent to A,
then $x\varphi$ is adherent to $A\varphi$

- if $A \subset X$, then $\overline{A}\varphi \subset \overline{A\varphi}$

- if y is a point of Y,
if B is a subset of Y, and
if y is interior to B,
then $y\varphi^{-1}$ is interior to $B\varphi^{-1}$

- if $B \subset Y$, then $\overset{o}{B}\varphi^{-1} \subset \left(B\varphi^{-1}\right)^{o}$

- if F is a filter on X,

if x is a point of X, and

if $F \to x$,

then

$F\varphi \to x\varphi$

- if $\mathcal{F}$ is a filterbase on X,

if x is a point of X, and

if $\mathcal{F} \to x$,

then

$\mathcal{F}\varphi \to x\varphi$

- if $(x_i \mid i \in I)$ is a net in X,

if x is a point of X, and

if $(x_i \mid i \in I) \to x$,

then

$(x_i\varphi \mid i \in I) \to x\varphi$

D. local properties of topological spaces

• suppose that a property called
admissible
is meaningful for top sps;
now every subset of a top sp is again a top sp
in a canonical (= uniquely defined) way;
(note this topological situation is in strong contrast
to the usual algebraic situation eg
it is in the nature of groups that
to be a subgroup of a group is indeed a strong property);
thus it is meaningful to speak of
admissible subsets of a top sp;
a top sp X is locally admissible
$=_{df}$ if $x \in X$ and if U is a nbd of x,
then there exists an admissible nbd V of x st
$V \subset U$
= for every point x of X
the class of all admissble nbds of x
is a base of the nbd filter of x

Theorem (Hahn-Mazurkiewicz Theorem).
A Hausdorff space X is a topological curve,
that is, a continuous image of the closed unit interval,
iff X is nonempty compact connected locally connected
second-countable/metrizable.

• bioline
Hans Hahn
1879-1934
Austrian

• bioline
Stefan Mazurkiewicz
1888-1945
Polish

C. normal spaces
are intended to be
those topological spaces that
carry sufficiently many
real - valued continuous functions
to separate disjoint closed sets
(0 on one & 1 on the other);
the initial definition of normal space however
is conceptually simple
and
avoids the mention of real number;
its equivalence with this property is far from obvious

D. normal spaces

let

X $\in$ top sp

then

X is a normal topological space

= X is a normal space

= X is normal

$=_{df}$ any one of the following

pairwise equivalent statements

is satisfied

(1) if A and B are closed subsets of X st

$A \cap B = \varnothing$

then there exist open subsets C and D of X st

$A \subset C$

$B \subset D$

$C \cap D = \varnothing$

(1') if A and B are open subsets of X st

$A \cup B = X$

then there exist closed subsets C and D of X st

$A \supset C$

$B \supset D$

$C \cup D = X$

(2) if $(A_i \mid i \in I)$ is a finite family
of closed subsets of X st

$$\bigcap_{i \in I} A_i = \varnothing$$

then there exists a family $(B_i \mid i \in I)$
of open subsets of X st

$$A_i \subset B_i \qquad (\forall\, i \in I)$$

$$\bigcap_{i \in I} B_i = \varnothing$$

(2') if $(A_i \mid i \in I)$ is a finite family
of open subsets of X st

$$\bigcup_{i \in I} A_i = X$$

then there exists a family $(B_i \mid i \in I)$
of closed subsets of X st

$$A_i \supset B_i \qquad (\forall\, i \in I)$$

$$\bigcup_{i \in I} B_i = X$$

(3) if $\left(A_i \mid i \in I\right)$ is a finite family
of closed subsets of X,
then there exists a family $\left(B_i \mid i \in I\right)$
of open subsets of X st
$A_i \subset B_i \quad (\forall i \in I)$
&

$$\forall J \subset I. \bigcap_{j \in J} A_j = \varnothing \Leftrightarrow \bigcap_{j \in J} B_j = \varnothing$$

wiet

$$\forall J \subset I. \bigcap_{j \in J} A_j \neq \varnothing \Leftrightarrow \bigcap_{j \in J} B_j \neq \varnothing$$

(3′) if $\left(A_i \mid i \in I\right)$ is a finite family

of open subsets of X,

then there exists a family $\left(B_i \mid i \in I\right)$

of closed subsets of X st

$A_i \supset B_i \quad (\forall i \in I)$

&

$$\forall J \subset I. \bigcup_{j \in J} A_j = X \Leftrightarrow \bigcup_{j \in J} B_j = X$$

wiet

$$\forall J \subset I. \bigcup_{j \in J} A_j \neq X \Leftrightarrow \bigcup_{j \in J} B_j \neq X$$

C. verbal paraphrases
of the preceding properties

(1) disjoint closed sets
have disjoint neighborhoods

(1') every binary open cover
is shrinkable to a closed cover

(2) every finite disjoint family of closed sets
has disjoint neighborhoods

(2') every finite open cover
is shrinkable to a closed cover

(3) every finite closed cluster
is expandable to an open cluster
with the same nerve
(= intersection pattern of members)