The Perfect Definition of a Topology of a Set

#70 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
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Infinite Vistas Press
PVD RI
2002

GG70-1 (14)

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GG70-2

D. topology of a set

• five pairwise equivalent definitions of
a topology of a set
are given below;
each type of topology
canonically induces
each of the other four types of topology;
each closed chain of canonical inducements
leads back to the original topology

• an open-set topology
&
a closed-set topology
are complement-dual notions
& thus equivalent

• an interior topology
&
a closure topology
are complement-dual notions
& thus equivalent

• the complement-dual of
a neighborhood topology
is seemingly unuseful
because the complement-dual of
a point or rather a singleton set
is not a notion that in the past
has appeared to be helpful

• the shortest (?) definition:
a topology of a set
is defined to be
a cluster on the set that is closed under
finite intersection & arbitrary union

• a topological space
is defined to be
a set provided with a topology of the set

• the elements of a topological space
are called
points of the space

• it is to be suggested in a philosophical vein that
a geometry
is anything that has
a topological space associated with it
so that
space
means primarily
topological space

D. open - set topology

let

• X $\in$ set

then

• an open - set topology

for / in / <u>of</u> / on / over X

$=_{df}$ a cluster $\mathcal{A}$ on X

st

(1) $\mathcal{A}$ is closed under finite intersection

ie

if $\mathcal{E}$ is a finite subclass of $\mathcal{A}$, then $\bigcap \mathcal{E} \in \mathcal{A}$

(2) $\mathcal{A}$ is closed under arbitrary union

ie

if $\mathcal{E}$ is a subclass of $\mathcal{A}$, then $\bigcap \mathcal{E} \in \mathcal{A}$

wh

the members of $\mathcal{A}$ are called

open sets

D. closed - set topology

let

• X $\in$ set

then

• a closed - set topology

for / in / <u>of</u> / on / over X

$=_{df}$ a cluster $\mathcal{A}$ on X

st

(1) $\mathcal{A}$ is closed under finite union

ie

if $\mathcal{E}$ is a finite subclass of $\mathcal{A}$, then $\bigcup \mathcal{E} \in \mathcal{A}$

(2) $\mathcal{A}$ is closed under arbitrary intersection

ie

if $\mathcal{E}$ is a subclass of $\mathcal{A}$, then $\bigcap \mathcal{E} \in \mathcal{A}$

wh

the members of $\mathcal{A}$ are called

closed sets

D. interior topology

let

• $X \in$ set

then

• an interior topology

for / in / <u>of</u> / on / over X

$=_{df}$ a function on PX to PX

$=_{cl}$ the interior operator

$=_{dn}$ int: $PX \to PX$ wh int ← <u>int</u>erior

$$A \mapsto \text{int}\,A = \overset{o}{A} = A^{o}$$

$$=_{rd} \text{ A interior } = \text{ A ope } =_{cl} \text{ the interior of A}$$

st

(1) int is intersection - preserving

ie

$\text{int}\,(A \cap B) = \text{int}\,A \cap \text{int}\,B \quad (A, B \in PX)$

(2) int is contractive

ie

$\text{int}\,A \subset A \quad (A \in PX)$

(3) int is space - preserving

ie

$\text{int}\,X = X$

D. closure topology

let

• X $\in$ set

then

• a closure topology

for / in / of / on / overX

$=_{df}$ a function on PX to PX

$=_{cl}$ the closure operator

$=_{dn}$ cls: PX $\to$ PX wh cls $\leftarrow$ closure

$A \mapsto \text{cls } A = \overline{A} = A^{-}$

$=_{rd}$ A closure = A bar $=_{cl}$ the closure of A

st

(1) cls is union - preserving

ie

$\text{cls}(A \cup B) = \text{cls} A \cup \text{cls} B \quad (A, B \in PX)$

(2) cls is expansive

ie

$\text{cls} A \supset A \quad (A \in PX)$

(3) cls is empty - set - preserving

ie

$\text{cls } \varnothing = \varnothing$

D. neighborhood topology

let

• X $\in$ set

then

• a neighborhood topology

for / in / <u>of</u> / on / over X

$=_{df}$ a family $(\mathcal{N}_x | x \in X)$ of filters on X

with X as index set

st

(1) $x \in \bigcap \mathcal{N}_x \quad (x \in X)$

(2) $\forall x \in X . \forall U \in \mathcal{N}_x . \exists V \in \mathcal{N}_x . \forall y \in V . U \in \mathcal{N}_y$

wh

the members of $\mathcal{N}_x$ $(x \in X)$ are called

neighborhoods of x

D. basic notions in an open - set topological space

let

• X $\in$ open - set topological space

then

• a subset A of X is closed

$=_{df}$ the complement of A is open

• the interior of a subset A of X

$=_{df}$ the union of all open sets contained in A

= the greatest open set contained in A

• the closure of a subset A of X

$=_{df}$ the intersection of all closed sets containing A

= the least closed set containing A

• a neighborhood of a point x of X

$=_{df}$ a subset of X

that contains an open set that contains x

D. basic notions in a closed - set topological space

let

• X $\in$ closed - set topological space

then

• a subset A of X is open

$=_{df}$ the complement of A is closed

• the interior of a subset A of X

$=_{df}$ the union of all open sets contained in A

= the greatest open set contained in A

• the closure of a subset A of X

$=_{df}$ the intersection of all closed sets containing A

= the least closed set containing A

• a neighborhood of a point x of X

$=_{df}$ a subset of X

that contains an open set that contains x

D. basic notions in an interior topological space

let

• X $\in$ interior topological space

then

• a subset A of X is open

$=_{df}$ the interior of A equals A

• a subset A of X is closed

$=_{df}$ the complement of A is open

• the closure of a subset A of X

$=_{df}$ the complement of the interior of the complement of A

• a neighborhood of a point x of X

$=_{df}$ a subset of X that contains an open set that contains x

D. basic notions in a closure topological space

let

• X $\in$ closure topological space

then

• a subset A of X is closed

$=_{df}$ the closure of A equals A

• a subset A of X is open

$=_{df}$ the complement of A is closed

• the interior of a subset A of X

$=_{df}$ the complement of the closure of the complement of A

• a neighborhood of a point x of X

$=_{df}$ a subset of X that contains an open set that contains x

D. basic notions in a neighborhood topological space

let

• X $\in$ neighborhood topological space

then

• a subset A of X is open

$=_{df}$ A is a neighborhood of every point of A

• a subset A of X is closed

$=_{df}$ the complement of A is open

• the interior of a subset A of X

$=_{df}$ the union of all open sets contained in A

= the greatest open set contained in A

• the closure of a subset A of X

$=_{df}$ the intersection of all closed sets containing A

= the least closed set containing A