A Smidgen of Symmetric Functions

#68 of Gottschalk's Gestalts

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GG68-2

☐ symmetric functions
are here presented
in a formal syntactic way
ito variables & polynomials & expressions
rather than in an equivalent logically precise way
using inp the notion of function as set;
the justification of this classical approach is
that it is clear
how the set-theoretic description proceeds
and
that the syntactic language is
more picturesque & colorful
& often easier to use & understand;
the syntactic language
may help in making the formulas/functions
more vivid & visual & visualizable & tactile

Standing Notation.

• for the sake of simpler hypotheses
in an initial exposition of symmetric functions,
let the field of complex numbers be
the algebraic base structure;
all considerations will be relative to
the complex number field;
it will be clear afterward (more or less)
what algebraic structures will support
what algebraic notions
and
what algebraic arguments

• let $n \in$ positive integer

• let $x_1, x_2, \cdots, x_n$ be n independent
complex variables

• other notation: for any positive integer m
the m - file $\underline{m} =_{df} \{1, 2, 3, \cdots, m\}$
the m - seg $\hat{m} =_{df} \{0, 1, 2, 3, \cdots, m\}$
wh seg ← <u>seg</u>ment

D. the $n+1$ elementary symmetric functions
of n variables

let

• $k \in \hat{n} = \{0, 1, 2, \cdots, n\}$

then

• the elementary symmetric function of degree k
for / in / of / on $x_1, x_2, \cdots, x_n$

= the kth elementary symmetric function
for / in / of / on $x_1, x_2, \cdots, x_n$

$=_{dn}$ $\sigma_k(x_1, x_2, \cdots, x_n)$

$=_{rd}$ sig (ma) (sub) k of $x_1, x_2, \cdots, x_n$

$=_{ab}$ σ_k

$=_{rd}$ sig (ma) (sub) k

$=_{df}$ the polynomial of degree k in $x_1, x_2, \cdots, x_n$
which is the sum of all $\binom{n}{k}$ products
of $x_1, x_2, \cdots, x_n$ taken k at a time

☐ examples

- $n = 1$

$\sigma_0 = 1$

$\sigma_1 = x_1$

- $n = 2$

$\sigma_0 = 1$

$\sigma_1 = x_1 + x_2$

$\sigma_2 = x_1 x_2$

- $n = 3$

$\sigma_0 = 1$

$\sigma_1 = x_1 + x_2 + x_3$

$\sigma_2 = x_1 x_2 + x_1 x_3 + x_2 x_3$

$\sigma_3 = x_1 x_2 x_3$

- $n = 4$

$\sigma_0 = 1$

$\sigma_1 = x_1 + x_2 + x_3 + x_4$

$\sigma_2 = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4$

$\sigma_3 = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4$

$\sigma_4 = x_1 x_2 x_3 x_4$

☐ in the expression for an elementary symmetric function
the canonical order
of the variables in a term
and
of the terms themselves
is taken to be
the lexicographic order of variables/terms
as is illustrated above
or, what amounts to the same thing here,
the numerical order of the subscripts

☐ the lowercase Greek letter sigma σ with subscripts
may have been chosen
to denote the elementary symmetric functions
because
s is the initial letter of both
the word 'symmetric' & the word 'sum',
the elementary symmetric functions being
certain kinds of sums,
and because
the Latin/English letter ess S s
is the phonetic equivalent and the transliteration of
the Greek letter sigma Σ σ ;
as a further notational comment
the subscript on sigma
matches
the number of factors
in each term of the polynomial

□ the elementary symmetric functions
are polynomials
& are also called
elementary symmetric polynomials;
but in usage
the word 'function' may nudge out
the word 'polynomial' here
on the grounds of history
and also
on the grounds of brevity
because 'function'
has 2 syllables and 8 letters
whereas 'polynomial' has
5 syllables and 10 letters

☐ some summation - index notation
for the symmetic functions

- $\sigma_1 = \sum_{i=1}^{n} x_i = \sum_{i\in \underline{n}} x_i = \sum_{1\le i\le n} x_i = \sum_{i} x_i$

- $\sigma_2 = \sum_{\substack{i,j=1\\ i<j}}^{n} x_i x_j = \sum_{\substack{i,j\in \underline{n}\\ i<j}} x_i x_j = \sum_{1\le i<j\le n} x_i x_j = \sum_{i<j} x_i x_j$

- $\sigma_3 = \sum_{\substack{i,j,k=1\\ i<j<k}}^{n} x_i x_j x_k = \sum_{\substack{i,j,k\in \underline{n}\\ i<j<k}} x_i x_j x_k = \sum_{1\le i<j<k\le n} x_i x_j x_k$

$$= \sum_{i<j<k} x_i x_j x_k$$

etc

• σ_k

$$= \sum_{\substack{i_1,\cdots,i_k=1 \\ i_1<\cdots<i_k}}^{n} x_{i_1} \cdots x_{i_k}$$

$$= \sum_{\substack{i_1,\cdots,i_k \in \underline{n} \\ i_1<\cdots<i_k}} x_{i_1} \cdots x_{i_k}$$

$$= \sum_{1\le i_1<\cdots<i_k \le n} x_{i_1} \cdots x_{i_k}$$

$$= \sum_{i_1<\cdots<i_k} x_{i_1} \cdots x_{i_k}$$

wh $k \in \underline{n}$

• also

$$\sigma_n = \prod_{i=1}^{n} x_i = \prod_{i\in\underline{n}} x_i = \prod_{1\le i\le n} x_i = \prod_{i} x_i$$

☐ the generating function
for the elementary symmetric functions
is

• $(1+x_1 t)(1+x_2 t) \cdots (1+x_n t)$ wh $t \in$ complex var

$$= \prod_{i=1}^{n} (1+x_i t)$$

$$= \sum_{i=0}^{n} \sigma_i t^i$$

$$= \sigma_0 + \sigma_1 t + \sigma_2 t^2 + \cdots + \sigma_n t^n$$

inp

- $n = 1 \Rightarrow$

$$1 + x_1 t$$

$$= \sigma_0 + \sigma_1 t$$

- $n = 2 \Rightarrow$

$$(1 + x_1 t)(1 + x_2 t)$$

$$= 1 + (x_1 + x_2)t + (x_1 x_2)t^2$$

$$= \sigma_0 + \sigma_1 t + \sigma_2 t^2$$

- $n = 3 \Rightarrow$

$$(1 + x_1 t)(1 + x_2 t)(1 + x_3 t)$$

$$= 1 + (x_1 + x_2 + x_3)t + (x_1 x_2 + x_1 x_3 + x_2 x_3)t^2 + (x_1 x_2 x_3)t^3$$

$$= \sigma_0 + \sigma_1 t + \sigma_2 t^2 + \sigma_3 t^3$$

D. the sequence of power - sum symmetric functions of n variables

let

- k ∈ nonnegative integer

then

- the power - sum symmetric function of degree k
  for / in / of / on $x_1, x_2, \cdots, x_n$

= the kth power - sum symmetric function
  for / in / of / on $x_1, x_2, \cdots, x_n$

$=_{dn}$ $s_k(x_1, x_2, \cdots, x_n)$

$=_{rd}$ ess (sub) k of $x_1, x_2, \cdots, x_n$

$=_{ab}$ s_k

$=_{rd}$ ess (sub) k

$=_{df}$ the polynomial of degree k in $x_1, x_2, \cdots, x_n$
which is the sum of the kth powers of $x_1, x_2, \cdots, x_n$

$= x_1{}^k + x_2{}^k + \cdots + x_n{}^k$

$$= \sum_{i=1}^{n} x_i{}^k$$

whence

$$s_0 = x_1^0 + x_2^0 + \cdots + x_n^0 = \sum_{i=1}^{n} x_i^0 = n$$

$$s_1 = x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i$$

$$s_2 = x_1^2 + x_2^2 + \cdots + x_n^2 = \sum_{i=1}^{n} x_i^2$$

$$s_3 = x_1^3 + x_2^3 + \cdots + x_n^3 = \sum_{i=1}^{n} x_i^3$$

etc

$$s_k = x_1^k + x_2^k + \cdots + x_n^k = \sum_{i=1}^{n} x_i^k$$

☐ the generating function

for the power - sum symmetric functions

is

- $$\frac{1}{1-x_1 t}+\frac{1}{1-x_2 t}+\cdots+\frac{1}{1-x_n t}$$ wh $t \in$ complex var

$$= \sum_{i=1}^{n} \frac{1}{1-x_i t}$$

$$= \sum_{i=0}^{\infty} s_i t^i$$

$$= s_0 + s_1 t + s_2 t^2 + \cdots$$

note that the simple geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

plays a role here

inp

• $n = 1 \Rightarrow$

$$\frac{1}{1 - x_1 t}$$

$$= 1 + x_1 t + {x_1}^2 t^2 + {x_1}^3 t^3 + \cdots$$

$$= s_0 + s_1 t + s_2 t^2 + s_s t^3 + \cdots$$

• $n = 2 \Rightarrow$

$$\frac{1}{1 - x_1 t} + \frac{1}{1 - x_2 t}$$

$$= (1 + x_1 t + {x_1}^2 t^2 + {x_1}^3 t^3 + \cdots) + (1 + x_2 t + {x_2}^2 t^2 + {x_1}^3 t^3 + \cdots)$$

$$= 2 + (x_1 + x_2) t + ({x_1}^2 + {x_2}^2) t^2 + ({x_1}^2 + {x_2}^2) t^3 + \cdots$$

$$= s_0 + s_1 t + s_2 t^2 + s_s t^3 + \cdots$$

D. a function $f(x_1, x_2, \cdots, x_n)$

of the $n \geq 2$ independent variables $x_1, x_2, \cdots, x_n$

and

which takes values in any set

is said to be

(totally) symmetric

provided that

the value of the function is invariant

under every permutation of the variables

or equivalently

the value of the function is invariant

under the transposition of every pair of variables eg

• $f(x_1, x_2) \in$ sym

$\Leftrightarrow$

$f(x_1, x_2) = f(x_2, x_1)$ for all $x_1, x_2 \in \mathbb{C}$

• $f(x_1, x_2, x_3) \in$ sym

$\Leftrightarrow$

$f(x_1, x_2, x_3)$

$= f(x_1, x_3, x_2)$

$= f(x_3, x_2, x_1)$

$= f(x_2, x_1, x_3)$

for all $x_1, x_2, x_3 \in \mathbb{C}$

and hence also

$f(x_1, x_2, x_3)$

$= f(x_1, x_3, x_2)$

$= f(x_3, x_2, x_1)$

$= f(x_2, x_1, x_3)$

$= f(x_2, x_3, x_1)$

$= f(x_3, x_2, x_1)$

for all $x_1, x_2, x_3 \in \mathbb{C}$

□ every polynomial in symmetric functions
is clearly again a symmetric function;
the elementary symmetric functions
are all symmetric functions;
so every polynomial in the elementary symmetric functions
is again a symmetric polynomial;
the following theorem states that the converse also holds;
thus the class of elementary symmetic functons is
a particularly important class of symmetric functions

T. the fundamental theorem on symmetric polynomials:
every symmetric polynomial
of n variables over the complex field
is uniquely expressible as
a polynomial in the elementary symmetric functions
of these n variables over the complex field

□ the power - sum symmetric functions
are all symmetric polynomials;
by the fundamental theorem on symmetric functions
every power - sum symmetric function
is uniquely expressible as
a polynomial in the elementary symmetric functions;
explicit examples are given below:

- $n = 1$

$\sigma_0 = 1$
$\sigma_1 = x_1$

$s_0 = 1$
$= \sigma_0$
$s_1 = x_1$
$= \sigma_1$
$s_2 = x_1^2$
$= \sigma_1^2$
$s_3 = x_1^3$
$= \sigma_1^3$
$s_4 = x_1^4$
$= \sigma_1^4$
$s_5 = x_1^5$
$= \sigma_1^5$
etc

• $n = 2$

$\sigma_0 = 1$

$\sigma_1 = x_1 + x_2$

$\sigma_2 = x_1 x_2$

$s_0 = 2$

$= 2\sigma_0$

$s_1 = x_1 + x_2$

$= \sigma_1$

$s_2 = x_1^2 + x_2^2$

$= \sigma_1^2 - 2\sigma_2$

$s_3 = x_1^3 + x_2^3$

$= \sigma_1^3 - 3\sigma_1\sigma_2$

$s_4 = x_1^4 + x_2^4$

$= \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2$

$s_5 = x_1^5 + x_2^5 + x_3^5$

$= \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2$

etc

- $n = 3$

$$\sigma_0 = 1$$
$$\sigma_1 = x_1 + x_2 + x_3$$
$$\sigma_2 = x_1x_2 + x_1x_3 + x_2x_3$$
$$\sigma_3 = x_1x_2x_3$$

$$s_0 = 3$$
$$= 3\sigma_0$$
$$s_1 = x_1 + x_2 + x_3$$
$$= \sigma_1$$
$$s_2 = x_1^2 + x_2^2 + x_3^2$$
$$= \sigma_1^2 - 2\sigma_2$$
$$s_3 = x_1^3 + x_2^3 + x_3^3$$
$$= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$$
$$s_4 = x_1^4 + x_2^4 + x_3^4$$
$$= \sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2$$
$$s_5 = x_1^5 + x_2^5 + x_3^5$$
$$= \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1^2\sigma_3 + 5\sigma_1\sigma_2^2 - 5\sigma_2\sigma_3$$

etc

• n = 4

$\sigma_0 = 1$

$\sigma_1 = x_1 + x_2 + x_3 + x_4$

$\sigma_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$

$\sigma_3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$

$\sigma_4 = x_1x_2x_3x_4$

$s_0 = 4$

$\quad = 4\sigma_0$

$s_1 = x_1 + x_2 + x_3 + x_4$

$\quad = \sigma_1$

$s_2 = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2$

$\quad = \sigma_1{}^2 - 2\sigma_2$

$s_3 = x_1{}^3 + x_2{}^3 + x_3{}^3 + x_4 3$

$\quad = \sigma_1{}^3 - 3\sigma_1\sigma_2 + 3\sigma_3$

$s_4 = x_1{}^4 + x_2{}^4 + x_3{}^4 + x_4{}^4$

$\quad = \sigma_1{}^4 - 4\sigma_1{}^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2{}^2 - 4\sigma_4$

$s_5 = x_1{}^5 + x_2{}^5 + x_3{}^5 + x_4{}^5$

$\quad = \sigma_1{}^5 - 5\sigma_1{}^3\sigma_2 + 5\sigma_1{}^2\sigma_3 + 5\sigma_1\sigma_2{}^2 - 5\sigma_1\sigma_4 - 5\sigma_2\sigma_3$

etc

note: think of

the formulas for the s_k ito the σ' s

as n takes on various values;

no matter what the value of n is,

the formula for s_0 is

$s_0 = n\sigma_0$;

no matter what the value of n is

the formula for s_1 is

$s_1 = \sigma_1$;

thinking of the value of $k \geq 2$ as fixed

and

thinking of the value of n as increasing from 1

in unit incremental steps,

the formula for s_k

adds on terms until n attains k

and then afterward stays the same;

defining $\sigma_k = 0$ for $k \in$ int & $k > n$,

the formulas for s_k with larger n

collapse to

the formulas for s_k with smaller n

if they change at all

☐ to see a pattern in the formulas
for the s's ito the σ's,
look at the determinant forms

$$s_2 = \begin{vmatrix} \sigma_1 & -1 \\ -2\sigma_2 & \sigma_1 \end{vmatrix}$$

$$s_3 = \begin{vmatrix} \sigma_1 & -1 & 0 \\ -2\sigma_2 & \sigma_1 & -1 \\ 3\sigma_3 & -\sigma_2 & \sigma_1 \end{vmatrix}$$

$$s_4 = \begin{vmatrix} \sigma_1 & -1 & 0 & 0 \\ -2\sigma_2 & \sigma_1 & -1 & 0 \\ 3\sigma_3 & -\sigma_2 & \sigma_1 & -1 \\ -4\sigma_4 & \sigma_3 & -\sigma_2 & \sigma_1 \end{vmatrix}$$

a general determinant can be written down
for the s' s ito the σ' s
but the alternation in signs
makes the notation rather unwieldly,
likely occupying a whole page
for the sake of attempted clarity;
later when the a' s are defined simply ito the σ' s,
a general determinant for the s' s ito the a' s
which is easier on the eyes
and more readily comprehensible
is written down

☐ just like the elementary symmetric functions,
the power - sum symmetric functions
are also capable of expressing
any symmetric polynomial
as a polynomial in these functions,
as witness the following theorem

T. every symmetric polynomial
of n variables over the complex field
is uniquely expressible as
a polynomial in the power - sum symmetric functions
of these n variables over the complex field

□ some examples of the σ's ito the s's

• $n = 1$

$$\sigma_0 = s_0$$

$$\sigma_1 = s_1$$

• $n = 2$

$$\sigma_0 = \frac{1}{2} s_0$$

$$\sigma_1 = s_1$$

$$\sigma_2 = \frac{1}{2}\left(s_1{}^2 - s_2\right)$$

• $n = 3$

$$\sigma_0 = \frac{1}{3} s_0$$

$$\sigma_1 = s_1$$

$$\sigma_2 = \frac{1}{2}\left(s_1{}^2 - s_2\right)$$

$$\sigma_3 = \frac{1}{6}\left(s_1{}^3 - 3s_1 s_2 + 2s_3\right)$$

• n = 4

$$\sigma_0 = \frac{1}{4} s_0$$

$$\sigma_1 = s_1$$

$$\sigma_2 = \frac{1}{2}\left(s_1{}^2 - s_2\right)$$

$$\sigma_3 = \frac{1}{6}\left(s_1{}^3 - 3s_1 s_2 + 2s_3\right)$$

$$\sigma_4 = \frac{1}{24}\left(s_1{}^4 - 6s_1{}^2 s_2 + 8s_1 s_3 + 3s_2{}^2 - 6s_4\right)$$

• n = 5

$$\sigma_0 = \frac{1}{5} s_0$$

$$\sigma_1 = s_1$$

$$\sigma_2 = \frac{1}{2}\left(s_1{}^2 - s_2\right)$$

$$\sigma_3 = \frac{1}{6}\left(s_1{}^3 - 3s_1 s_2 + 2s_3\right)$$

$$\sigma_4 = \frac{1}{24}\left(s_1{}^4 - 6s_1{}^2 s_2 + 8s_1 s_3 + 3s_2{}^2 - 6s_4\right)$$

$$\sigma_5 = \frac{1}{120}(s_1{}^5 - 10s_1{}^3 s_2 + 20s_1{}^2 s_3 + 15s_1 s_2{}^2 - 30s_1 s_4 - 20s_2 s_3 + 24s_5)$$

□ the following determinant expresses
the σ's ito the s's

$$\sigma_k = \frac{1}{k!}\begin{vmatrix} s_1 & 1 & 0 & 0 & 0\cdots 0 \\ s_2 & s_1 & 2 & 0 & 0\cdots 0 \\ s_3 & s_2 & s_1 & 3 & 0\cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_k & s_{k-1} & \cdots & \cdots & s_1 \end{vmatrix}$$

wh $k \in$ int & $1 \le k \le n$

□ canonical polynomial & canonical polynomial equation

the canonical polynomial $P(x)$
over the complex field
determined by $x_1, x_2, \cdots, x_n$ as zeros
&
the canonical polynomial equation $P(x) = 0$
over the complex field
determined by $x_1, x_2, \cdots, x_n$ as roots
are described below

let a_0 be an arbitrarily chosen nonzero complex number

&

let $x \in$ complex var

form the polynomial P(x)

& thence the polynomial equation $P(x) = 0$

as follows:

$$P(x)$$

$$= a_0(x - x_1)(x - x_2)\cdots(x - x_n) \quad (a_0 \neq 0)$$

$$= a_0 \prod_{k=1}^{n} (x - x_k)$$

$$= a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

$$= \sum_{k=0}^{n} a_k x^{n-k}$$

wh

$$a_0 = a_0\,\sigma_0$$

$$a_1 = -a_0\,\sigma_1$$

$$a_2 = a_0\,\sigma_2$$

$$a_3 = -a_0\,\sigma_3$$

$\vdots$

$$a_n = (-1)^n a_0\,\sigma_n$$

& in general

$$a_k = (-1)^k a_0\,\sigma_k \qquad (k \in \hat{n})$$

ift

$$\sigma_0 = 1$$

$$\sigma_1 = -\frac{a_1}{a_0}$$

$$\sigma_2 = \frac{a_2}{a_0}$$

$$\sigma_3 = -\frac{a_3}{a_0}$$

$$\vdots$$

$$\sigma_n = (-1)^n \frac{a_n}{a_0}$$

& in general

$$\sigma_k = (-1)^k \frac{a_k}{a_0} \qquad (k \in \hat{n})$$

it is clear that
the σ' s and the a' s
are essentially equivalent
&
the distinction between the σ' s and the a' s
is a slight notational difference
but this notational change
will make certain expressions / formulas easier
to view & handle ito of the a' s rather than the σ' s

□ the preceding several pages
may be paraphrased by stating that
for a polynomial equation in one variable
that factors completely
(as always the case in the complex field),
the coefficients are equal to
the alternatingly signed
elementary symmetric functions
of the roots
times
the leading coefficient

☐ the following determinant expresses
the a's ito the s's

$$a_k = \frac{(-1)^k}{k!} a_0 \begin{vmatrix} s_1 & 1 & 0 & 0 & 0 \cdots 0 \\ s_2 & s_1 & 2 & 0 & 0 \cdots 0 \\ s_3 & s_2 & s_1 & 3 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_k & s_{k-1} & \cdots & \cdots & s_1 \end{vmatrix}$$

wh $k \in$ int & $1 \leq k \leq n$

□ the following determinant
expresses the s's ito the a's

$$s_k = \frac{(-1)^k}{a_0{}^k} \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 \cdots 0 \\ 2a_2 & a_1 & a_0 & 0 & 0 \cdots 0 \\ 3a_3 & a_2 & a_1 & a_0 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ ka_k & a_{k-1} & \cdots & \cdots & a_1 \end{vmatrix}$$

wh $k \in$ int & $1 \le k \le n$

T. Newton' s identities (first form)

- $$\sum_{i=0}^{k-1} (-1)^i \sigma_i s_{k-i} + (-1)^k k\sigma_k = 0$$

wh $k \in$ int & $1 \le k \le n$

- $$\sum_{i=0}^{n} (-1)^i \sigma_i s_{k-i} = 0$$

wh $k \in$ int & $k \ge n$

ion

- $\sigma_0 s_k - \sigma_1 s_{k-1} + \cdots + (-1)^{k-1} \sigma_{k-1} s_1 + (-1)^k k\sigma_k \;=\; 0$

wh $k \in$ int & $1 \le k \le n$

- $\sigma_0 s_k - \sigma_1 s_{k-1} + \cdots + (-1)^n \sigma_n s_{k-n} \;=\; 0$

wh $k \in$ int & $k \ge n$

inp

(n, k)

(1, 1) $\sigma_0 s_1 - \sigma_1 = 0$

(2, 1) $\sigma_0 s_1 - \sigma_1 = 0$

(2, 2) $\sigma_0 s_2 - \sigma_1 s_1 + 2\sigma_2 = 0$

(3, 1) $\sigma_0 s_1 - \sigma_1 = 0$

(3, 2) $\sigma_0 s_2 - \sigma_1 s_1 + 2\sigma_2 = 0$

(3, 3) $\sigma_0 s_3 - \sigma_1 s_2 + \sigma_2 s_1 - 3a\sigma_3 = 0$

etc

(n, k)

(1, 1) $\sigma_0 s_1 - \sigma_1 s_0 = 0$

(1, 2) $\sigma_0 s_2 - \sigma_1 s_1 = 0$

(1, 3) $\sigma_0 s_3 - \sigma_1 s_2 = 0$

etc

(2, 2) $\sigma_0 s_2 - \sigma_1 s_1 + \sigma_2 s_0 = 0$

(2, 3) $\sigma_0 s_3 - \sigma_1 s_2 + \sigma_2 s_1 = 0$

(2, 4) $\sigma_0 s_4 - \sigma_1 s_3 + \sigma_2 s_2 = 0$

etc

(3, 3) $\sigma_0 s_3 - \sigma_1 s_2 + \sigma_2 s_1 - \sigma_3 s_0 = 0$

(3, 4) $\sigma_0 s_4 - \sigma_1 s_3 + \sigma_2 s_2 - \sigma_3 s_1 = 0$

(3, 5) $\sigma_0 s_5 - \sigma_1 s_4 + \sigma_2 s_3 - \sigma_3 s_2 = 0$

etc

etc

T. Newton's identities (second form)

- $\sum_{i=0}^{k-1} a_i s_{k-i} + k\, a_k = 0$

wh $k \in$ int & $1 \le k \le n$

- $\sum_{i=0}^{n} a_i s_{k-i} = 0$

wh $k \in$ int & $k \ge n$

ion

• $a_0 s_k + a_1 s_{k-1} + \cdots + a_{k-1} s_1 + k a_k \;=\; 0$
wh $k \in$ int & $1 \le k \le n$

• $a_0 s_k + a_1 s_{k-1} + \cdots + a_n s_{k-n} \;=\; 0$
wh $k \in$ int & $k \ge n$

inp

(n, k)

$(1,\ 1)\quad a_0 s_1 + a_1 = 0$

$(2,\ 1)\quad a_0 s_1 + a_1 = 0$

$(2,\ 2)\quad a_0 s_2 + a_1 s_1 + 2a_2 = 0$

$(3,\ 1)\quad a_0 s_1 + a_1 = 0$

$(3,\ 2)\quad a_0 s_2 + a_1 s_1 + 2a_2 = 0$

$(3,\ 3)\quad a_0 s_3 + a_1 s_2 + a_2 s_1 + 3a_3 = 0$

etc

(n, k)

(1, 1) $a_0 s_1 + a_1 s_0 = 0$
(1, 2) $a_0 s_2 + a_1 s_1 = 0$
(1, 3) $a_0 s_3 + a_1 s_2 = 0$
etc

(2, 2) $a_0 s_2 + a_1 s_1 + a_2 s_0 = 0$
(2, 3) $a_0 s_3 + a_1 s_2 + a_2 s_1 = 0$
(2, 4) $a_0 s_4 + a_1 s_3 + a_2 s_2 = 0$
etc

(3, 3) $a_0 s_3 + a_1 s_2 + a_2 s_1 + a_3 s_0 = 0$
(3, 4) $a_0 s_4 + a_1 s_3 + a_2 s_2 + a_3 s_1 = 0$
(3, 5) $a_0 s_5 + a_1 s_4 + a_2 s_3 + a_3 s_2 = 0$
etc

etc

☐ Newton's identities relate
the σ's & the a's on the one hand
with
the s's on the other;
thus there are two forms of Newton's identities;
one form consists of σ's & s's together
and
the other form consists a's & s's together;
each form is a sequence of formulas
depending on a pos int var k;
each form is expressed by two equations
because there is a change
in the structure of the first equation
that affects the last term
when the parameter k ∈ pos int var
changes in possible value
from weakly less than n
to weakly greater than n;

the situation of two equations for each form
is brought about
at least partly because
there are only finitely many σ' s & a' s
but there are infinitely many s' s;
the two forms are thoroughly equivalent
since they are only slight notational variants of each other;
because the second form does not contain
the powers of -1 and the minus signs
that the first form contains,
the second form is
more compact and neater in appearance

□ the first equation of the second form of Newton's identities is expressible as a matrix equation as follows:

$$\begin{bmatrix} 1 & & & & & \\ s_1 & 2 & & & & \\ s_2 & s_1 & 3 & & & \\ s_3 & s_2 & s_1 & 4 & & \\ \vdots & & & & & \\ s_{n-1} & s_{n-2} & \cdots & s_1 & n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = -a_0 \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ \vdots \\ s_n \end{bmatrix}$$

□ the second equation of the second form of Newton's identities
may be thought of as a sequence
of the inner product of two vectors
viz

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} s_k \\ s_{k-1} \\ s_{k-2} \\ \vdots \\ s_{k-n} \end{bmatrix} = 0 \quad \text{wh } k \in \text{int} \;\&\; k \geq n$$

D. a function $f(x_1, x_2, \cdots, x_n)$
of the n independent variables $x_1, x_2, \cdots, x_n$
and
which takes values in an additive group say
is said to be
alternating
provided that
the value of the function ' changes sign'
(ie changes to the negation)
under the transposition of every pair of variables

eg

• $f(x_1, x_2) \in$ alt

$\Leftrightarrow$

$f(x_1, x_2) = -f(x_2, x_1)$ for all $x_1, x_2 \in \mathbb{C}$

• $f(x_1, x_2, x_3) \in$ alt

$\Leftrightarrow$

$$\begin{aligned} & f(x_1, x_2, x_3) \\ = \ & -f(x_1, x_3, x_2) \\ = \ & -f(x_3, x_2, x_1) \\ = \ & -f(x_2, x_1, x_3) \end{aligned}$$

for all $x_1, x_2, x_3 \in \mathbb{C}$

and hence also

$$\begin{aligned} & f(x_1, x_2, x_3) \\ = \ & -f(x_1, x_3, x_2) \\ = \ & -f(x_3, x_2, x_1) \\ = \ & -f(x_2, x_1, x_3) \\ = \ & \ \ f(x_2, x_3, x_1) \\ = \ & \ \ f(x_3, x_1, x_2) \end{aligned}$$

for all $x_1, x_2, x_3 \in \mathbb{C}$

D. the primitive alternating function / polynomial

• the primitive alternating function / polynomial
of degree $n \geq 2$
for / in / of / on $x_1, x_2, \cdots, x_n$
$=_{dn}$ $A(x_1, x_2, \cdots, x_n)$
$=_{ab}$ A
$=_{df}$ the polynomial of degree n in $x_1, x_2, \cdots, x_n$
which is
the product of the differences $x_i - x_j$
of all $\binom{n}{2}$ pairs x_i, x_j ($i < j$ wh $i, j \in \underline{n}$)
$$= \prod_{\substack{i,j \in \underline{n} \\ i<j}} (x_i - x_j)$$

☐ examples

• $n = 2$

$A = x_1 - x_2$

• $n = 3$

$A = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

• $n = 4$

A

$= (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$

R. A is the simplest alternating polynomial in the x' s
& ∴
the square A^2 of A is a symmetric polynomial in the x' s;
hence A^2 is expressible
as a polynomial in the σ' s;
but a σ is plus - or - minus an a over a_0
so that A^2 is expressible as a polynomial
in the a' s over a_0;
this latter form of A^2
(with a factor to get rid of any denominator involving a_0)
is taken to be the discriminant
of the polynomial P(x)
and
of the polynomial equation P(x) = 0,
the discriminant becoming a polynomial
in the coefficients of P(x) viz the a' s

☐ the discriminant Δ
of the polynomial P(x)
and
of the polynomial equation P(x) = 0
is defined to be
the product of
a_0^{2n-2}
&
$$A^2 = \prod_{\substack{i,j \in \underline{n} \\ i<j}} (x_i - x_j)^2$$
whence
$$\Delta = a_0^{2n-2} A^2 = a_0^{2n-2} \prod_{\substack{i,j \in \underline{n} \\ i<j}} (x_i - x_j)^2$$

□ formulas for the discriminant

$$\Delta = a_0^{2n-2} \prod_{\substack{i,j \in \underline{n} \\ i<j}} (x_i - x_j)^2$$

$$= a_0^{2n-2} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

$$= a_0^{2n-2} \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ s_2 & s_3 & s_4 & \cdots & s_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_0} R(P, P')$$

wh $R(P,\ P')$ is the resultant
of P(x) and its derivative $P'(x)$;
the discriminant Δ is a symmetric function
of the roots $x_1,\ x_2,\ \cdots\ ,\ x_n$
and effectively
the square of their primitive alternating function;
the discriminant vanishes
iff
the equation $P(x)\ =\ 0$
has at least one multiple root;
the determinant on the x' s above
is called
the Vandermonde determinant
$V(x_1,\ x_2,\ \cdots\ ,\ x_n)$
of $x_1,\ x_2,\ \cdots\ ,\ x_n$;
it is a desideratum that the discriminant be
a polynomial in the coefficients;
it turns out that the discriminant is
a homogeneous polynomial in the coefficients
of degree $2n-2$

□ the capital Greek letter delta Δ
may have been chosen
to denote the discriminant
because
d is the initial letter of both
the word ' discriminant' & the word ' difference' ,
the discriminant being a product of differences,
and because
the Latin / English letter dee D d
is the phonetic equivalent and the transliteration of
the Greek letter delta Δ δ

☐ the discriminant of the quadratic equation

$ax^2 + bx + c = 0 \quad (a \neq 0)$

over the complex number field

is

$\Delta = b^2 - 4ac$

note: the discriminant Δ

is the radicand

in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the roots of the quadratic equation

☐ the discriminant of the cubic equation

$$ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0)$$

over the complex number field

is

$$\Delta = b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2$$

note: to see how this expression is related to Cardano's solution of the cubic equation, see below

☐ Cardano's formula for solving the cubic

the three roots of the cubic equation

$ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0)$

over the complex number field

are

$x = u + v$

wh

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}}$$

&

$$v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}}$$

&

the cube roots are chosen so that

their product is $-\frac{p}{3}$

&

(continued next page)

$$p = -\frac{r^2}{3} + s$$

$$q = \frac{2r^3}{27} - \frac{rs}{3} + t$$

$$r = \frac{b}{a}$$

$$s = \frac{c}{a}$$

$$t = \frac{d}{a}$$

GG68-64

note: in the Cardano formula

the radicand

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = \frac{p^3}{27} + \frac{q^2}{4}$$

appears twice under a square root sign

and $\sqrt{D}$ appears twice under a cube root sign;

now

D

$$= \frac{p^3}{27} + \frac{q^2}{4}$$

$$= -\frac{1}{108}\left(r^2 s^2 + 18rst - 4s^3 - 4r^3 t - 27t^2\right)$$

$$= -\frac{1}{108a^4}\left(b^2 c^2 + 18abcd - 4ac^3 - 4b^3 d - 27a^2 d^2\right)$$

$$= -\frac{1}{108a^4}\Delta$$

observe that

$$108 = 4 \times 27 = 2^2 \times 3^3$$

is the common denominator of

the original form of D

□ the discriminant Δ of the quartic equation

$ax^4 + bx^3 + cx^2 + dx + e = 0 \quad (a \neq 0)$

over the complex number field

is given by the expansion

of the seventh order determinant

$$\begin{vmatrix} 1 & b & c & d & e & 0 & 0 \\ 0 & a & b & c & d & e & 0 \\ 0 & 0 & a & b & c & d & e \\ 4 & 3b & 2c & d & 0 & 0 & 0 \\ 0 & 4a & 3b & 2c & d & 0 & 0 \\ 0 & 0 & 4a & 3b & 2c & d & 0 \\ 0 & 0 & 0 & 4a & 3b & 2c & d \end{vmatrix}$$

☐ bioline

Girolamo Cardano (Italian form of name)

Jerome Cardan (English / French form of name)

Hieronymus Cardanus (Latin form of name)

1501 - 1576

Italian

mathematician, astrologer, astronomer,

philosopher, physician, physicist