A Small Slice of Pi

#61 of Gottschalk's Gestalts

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by Walter Gottschalk

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☐ a small slice of pi
= a modicum of information about
everybody's favorite
transcendental number π

• the circle ratio
= the circumference-to-diameter ratio
of any euclidean circle
= the dimensionless periphery
of a euclidean circle with unit diameter
= pi
= π
= the lowercase Greek letter
corresponding to the English letter p
= the initial letter of the Greek word
περιφερεια
= periphery, circumference
from
περιφερειν (Greek)
= to carry around
from
περι (Greek)
= around
+
φερειν (Greek)
= to carry

• the symbol π for the circle ratio
was introduced in 1706 by William Jones
& was adopted in 1737 by Euler

• William Jones
1675-1749
English
mathematics textbook writer

• Leonhard Euler
1707-1783
Swiss, lived many years in Germany & Russia
algebraist, analyst, geometer, number theorist,
probabilist, applied mathematician, calculating prodigy;
most prolific mathematician of all time

• in 1882 Lindemann proved that π is transcendental

• Carl Louis Ferdinand von Lindemann
1852-1939
German
analyst, geometer

• ¿ what is your favorite transcendental number ?
i prefer pi (a palindrome)

• some approximations to π
of historical interest
are given below

• the Babylonian value of π from ca 2000 BCE

$$\pi \approx \frac{5^2}{2^3} = \frac{25}{8} = 3\frac{1}{8} = 3.125$$

which is accurate to one decimal place
before and after rounding off

• the Egyptian value of π from ca 2000 BCE

$$\pi \approx 4\left(\frac{8}{9}\right)^2 = \left(\frac{16}{9}\right)^2 = \left(\frac{4}{3}\right)^4 = \frac{2^8}{3^4} = \frac{256}{81} = 3\frac{13}{81} = 3.16\cdots$$

which is accurate to one decimal place
before rounding off

• the biblical value of π from ca 550 BCE

$\pi \approx 3$

which is
  the floor of π
= the integer part of π
= the nearest integer to π

•1 Kings 7:23 KJV
And he made a molten sea,
ten cubits from the one brim to the other:
it was round all about, and his height was five cubits:
and a line of thirty cubits did compass it round about.

• 2 Chronicles 4:2 KJV
Also he made a molten sea of ten cubits from brim to brim,
round in compass, and five cubits the height thereof;
and a line of thirty cubits did compass it round about.

• cubit
= ancient Middle Eastern standard unit of length
= length of forearm
  from elbow to tip of extended middle finger
= usually ca 18 inches
comes from the Latin word cubitum = elbow

• the Archimedean value of π from ca 250 BCE

$$\pi \approx \frac{22}{7} = 3\frac{1}{7} = 3.1428\cdots$$

which is accurate to two decimal places
before and after rounding off;
Archimedes was the first to devise a method
of theoretically calculating π to any degree of accuracy
by considering the lengths
of regular inscribed/circumscribed polygons
in/about a circle;
he used a 96-gon to obtain his value

• Archimedes of Syracuse
ca 287-212 BCE
Greek
mathematician, physicist, inventor;
Archimedes, Newton, Gauss (in chronological order)
are said to be
the three greatest mathematicians of all time

• the Ptolemaic value of π from ca 150 CE

$$\pi \approx \frac{377}{120} = 3\frac{17}{120} = 3.14166\cdots$$

which is accurate to four decimal places
before rounding off

• $\pi \approx 3.1416$
is called
the Ptolemaic decimal value of π

• Ptolemy of Alexandria = Claudius Ptolemaeus (Latin)
ca 85-ca 165 CE
Alexandrian
mathematician, astronomer, geographer

• the Chinese value of π from ca 470 CE

$$\pi \approx \frac{355}{113} = 3\frac{16}{113} = 3.1415929\cdots$$

which is accurate to six decimal places
before rounding off;
used by Tsu Chung-chih

• Tsu Chung-chih
430-501 CE
Chinese
mathematician

• note the numerical curiosity

$$\frac{355}{113} = \frac{377 - 22}{120 - 7}$$

which relates the three values of π

$$\frac{355}{113} = \text{ the Chinese value}$$

$$\frac{377}{120} = \text{ the Ptolemaic value}$$

$$\frac{22}{7} = \text{ the Archimedean value}$$

• the Indian value of π from ca 500 CE

$$\pi \approx 3.1416$$

which is accurate to four decimal places;
used by Aryabhata

• Aryabhata
ca 476-ca 550 CE
Indian (India)
algebraist, geometer, astronomer

• Brahmagupta's values of π from ca 628 CE
the 'practical value' of π is 3
&
the 'neat value' of π is $\sqrt{10} = 3.16\cdots$

• Brahmagupta
ca 588-660 CE
Indian (India)
algebraist, diophantine analyst, geometer, astronomer

• a Ramanujan value of π

$$\pi \approx \sqrt{\sqrt{\frac{2143}{22}}} = 3.14159265\cdots$$

which is accurate to eight decimal places

• Srinivasa Aiyangar Ramanujan
1887-1920
Indian (India)
analyst, number theorist;
self-taught, arrived at results by intuition

• the first infinite series ever found for π
is the simple and pretty
Leibniz series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

but to calculate π from this series
is practically impossible because of
the very slow convergence of the series;
eg 300 terms do not give an accuracy to
two decimal places
& 100,000 terms are needed for accuracy to
five decimal places

• Gottfried Wilhelm Leibniz
1646-1716
German
algebraist, analyst, logician, philosopher,
scholarly writer, diplomat, theologian;
independent codiscoverer with Newton
of the differential & integral calculus

• a more efficient way to calculate π
that is of historical interest
is to use Gregory's series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (-1 \le x \le 1)$$

and Machin's formula

$$\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$$

which was done in 1706 by Machin
to calculate π to 100 decimal places

• James Gregory
1638-1675
Scottish
analyst, geometer, inventor, physicist

• John Machin
1680-1751
English
computer of π, astronomer

• a series due to Euler whose sum involves π
(he had many such)

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} = \zeta(2)$$

where

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \quad (z \in \text{complex var})$$

is the zeta function of Riemann

• Georg Friedrich Bernhard Riemann
1826-1866
German
analyst, geometer, number theorist,
topologist, physicist

• Newton's series for π

in the series

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots \quad (-1 \le x \le 1)$$

substitute x = 1/2

• Isaac Newton
1641-1727
English
algebraist, analyst, geometer, astronomner, physicist, government official; independent codiscoverer with Leibniz of the differential & integral calculus;
Archimedes, Newton, Gauss (in chronological order)
are said to be
the three greatest mathematicians of all time

• in 1593 Viète gave
the first numerically precise infinite expression for π
viz
Viète's infinite product for π

$$\frac{2}{\pi} = \prod_{n=2}^{\infty} \cos\frac{\pi}{2^n} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

note that
to pass from any term of the product to the next term:

replace the last $\frac{1}{2}$ by $\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}$

• François Viète (French) = Franciscus Vieta (Latin)
1540-1603
French
algebraist, geometer, cryptanalyst, lawyer, statesman

• Wallis's infinite product for π

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots$$

• John Wallis
1616-1703
English
algebraist, analyst, geometer, logician,
historian of mathematics, calculating prodigy,
cryptanalyst, linguist, theologian,
royal chaplain to Charles II

• Brouncker's continued fraction for π

$$\frac{\pi}{4} = \frac{1}{1+}\frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2+}\cdots$$

• William Brouncker
1620-1684
Irish
analyst;
founding member & first president
of the Royal Society;
2nd Viscount Brouncker of Castle Lyons

• Lambert's regular continued fraction for π

the first few terms are

$$\pi = 3 + \frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\cdots$$

no law of formation is known;
the first ten convergents are:

(1)	3	:	1
(2)	22	:	7
(3)	333	:	106
(4)	355	:	113
(5)	103993	:	33102
(6)	104348	:	33215
(7)	208341	:	66317
(8)	312689	:	99532
(9)	833719	:	265381
(10)	1146408	:	364913

• Johann Heinrich Lambert
1728-1777
German
analyst, geometer, number theorist, probabilist,
applied mathematician, astronomer,
philosopher, physicist

• lots of definite integrals have values involving π;
here is a handful

(1) $$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$

which is likely the simplest integral for π
&
which could serve as
a simple closed analytic definition of π

(2) $$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$$

(3) $$\int_0^1 \frac{\log x}{x-1}\,dx = \frac{\pi^2}{6}$$

(4) the probability integral
= the area under the probability curve $y = e^{-x^2}$
$$= \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

(5) $$\int_0^\infty \frac{dx}{\sqrt{x}\,e^x} = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(6) $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

(7) $\int_0^\infty \frac{x}{\sinh x} dx = \frac{\pi^2}{4}$

(8) $\int_0^\pi \sin x \, dx = 2$

(9) $\int_0^{2\pi} \sin^2 x \, dx = \pi$

(10) $\int_0^{\frac{\pi}{4}} \log(1 + \tan x) \, dx = \frac{\pi}{8} \log 2$

• sometimes geometry is simpler than analysis
eg consider the geometric interpretation
of the following integral whose value is
not immediately obvious from analytical considerations:

$$\int_0^1 \sqrt{1 - x^2} \, dx = \frac{\pi}{4}$$

• from ancient times
some geometry involving π

consider
a circle
in the euclidean plane with

radius	r
diameter	d
circumference	C
area	A

then

$$C = 2\pi r = \pi d$$

$$A = \pi r^2 = \frac{1}{4}\pi d^2$$

consider a sphere
in euclidean 3-space
with
radius r
diameter d
surface area S
volume V

then

$$S = 4\pi r^2 = \pi d^2$$

$$V = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$$

• from modern times
some geometry involving π

consider
a closed regular surface S
in euclidean 3-space with
Gaussian curvature K
Euler characteristic χ
element of surface area $d\sigma$

then

$$\int_S K\, d\sigma = 2\pi\chi$$

this is the famous Gauss-Bonnet theorem/formula
which says
geometry = topology

• Carl Friedrich Gauss
1777-1855
German
algebraist, analyst, geometer, number theorist,
numerical analyst, probabilist, statistician, astronomer,
physicist, calculating prodigy;
Archimedes, Newton, Gauss (in chronological order)
are said to be the three greatest mathematicians of all time

• Pierre Ossian Bonnet
1819-1892
French
analyst, geometer

• from trigonometry
plane/solid angle measurement involves π

one round plane angle

= three hundred sixty degrees

= 360^o

= two pi radians

= $2\pi^r$

one round solid angle

= four pi steradians

= $4\pi^{sr}$

• from analysis
two Euler formulas involving π

Euler's little formula
unites
four basic constants
in one epiphany equation

$$e^{\pi i} + 1 = 0$$

which is an immediate consequence of
Euler's big formula

$$e^{ix} = \cos x + i \sin x$$

• from the theory of numbers
an asymptotic formula involving π

Stirling's formula
(which is actually due to De Moivre)
approximates large factorials

$$n! \sim \sqrt{2\pi e}\left(\frac{n}{e}\right)^{n+\frac{1}{2}} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} = \sqrt{2\pi n}\; e^{-n}\, n^{n}$$

as $n \to \infty$

where $n \in$ pos int var

• James Stirling
1692-1770
Scottish
analyst, industrial manager

• Abraham De Moivre
1667-1754
French-English
analyst, probabilist, statistician

• from the theory of probability
a geometrical probabilty involving π

the needle problem/theorem of Buffon

let a horizontal plane be ruled by
equi-spaced parallel straight lines
of distance d apart
&
let a uniform homogeneous straight needle
of length $L < d$ be dropped at random onto the plane

then
the probability p
that the needle will cross one of the lines
is

$$p = \frac{2L}{\pi d}$$

• Georges-Louis Leclerc, Comte de Buffon
1707-1788
French
probabilist, naturalist

• from quantum theory
& particle physics

π occurs in the statement of
the Heisenberg
indeterminacy/uncertainty
principle/relations

$$\Delta p\,\Delta q \geq \frac{h}{4\pi} \quad \& \quad \Delta E\,\Delta t \geq \frac{h}{4\pi}$$

where
p = momentum
q = position
E = energy
t = time

h = Planck’s constant
= the (elementary) quantum of action
= the ratio of the energy of a photon to its frequency

Δ = the uncertainty in the measurement of

• Werner Karl Heisenberg
1901-1976
German
theoretical physicist;
Nobel laureate for physics (1932)

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• π to 100 decimal places is given below;
note that the first zero occurs in the 32nd decimal place

π = 3.

14159 26535 89793 23846

26433 83279 50288 41971

69399 37510 58209 74944

59230 78164 06286 20899

86280 34825 34211 70679

…

• π has been calculated (1999) to
over two hundred six billion decimal places
(206,158,430,000 more precisely);
this numerical feat provides a test for large computers
and the opportunity of studying
the statistcal distribution of digits
in the decimal expansion of π, a transcendental number;
there is also the strong likelihood
that techniques will be discovered
that are useful in other contexts;
it is unknown (2000)
whether π has the ‘expected’
10% distribution of each digit;
so far the statistics look like it
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• mnemonic for π to thirty-one decimal places,
up to the first zero,
which uses letter counts of the words

π = 3.14159 26535 89793 23846 26433 83279 5+

How I want a drink, alcoholic of course,
after the heavy lectures involving quantum mechanics!
All of thy geometry, Herr Planck, is fairly hard.
You too struggle? Yes, we acquire knowledge daily.

the first sentence is due to
Sir James Hopwood Jeans