Geometry Shards

#59 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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GG59-1 (30)

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□ how to number

the 2^n – ants

wh $n \in$ pos int

Δ it is customary to denote

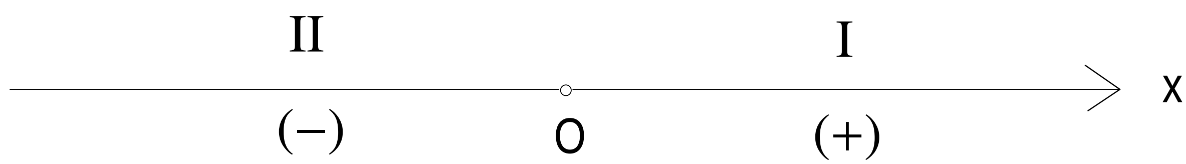
- halflines
- quadrants
- octants

etc

by capital Roman numerals

Δ the line with a cartesian coordinate system
is separated by the origin into
$2^1 = 2$ halflines
according to coordinate sign
as pictured & described below:

• the line with a cartesian coordinate system

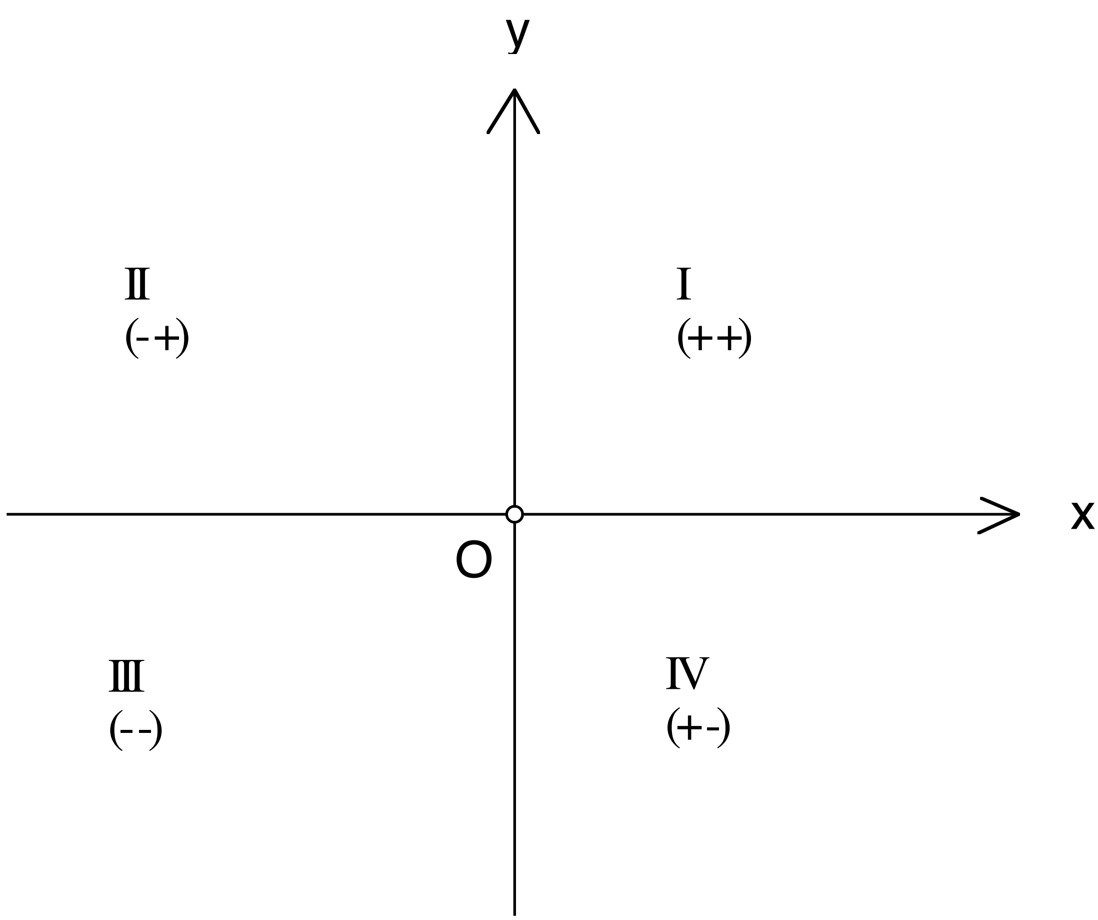


• the 2 halflines	the pattern of coordinate sign in the halflines
the 1st halfline = dn I	(+)
the 2nd halfline = dn II	(-)

• number the halflines
from positive to negative
coordinate sign

∆ the plane with a rectangular coordinate sytem
is separated by the 2 coordinate axes into
$2^2 = 4$ quadrants
according to coordinate signs
as pictured & described below:

• the plane with a rectangular coordinate system



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• the 4 quadrants

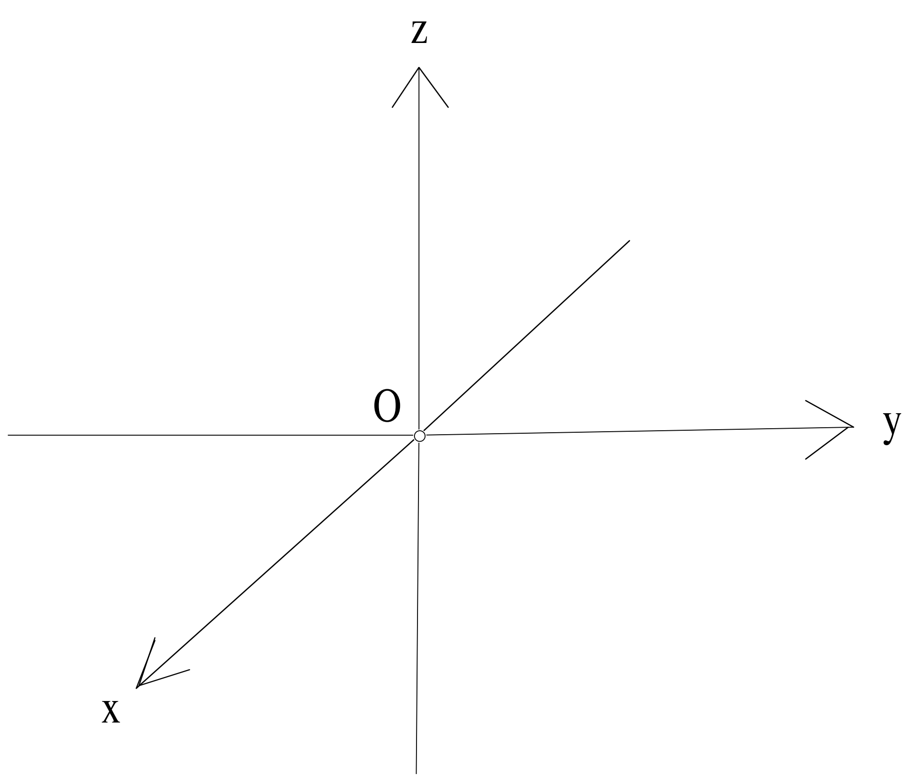
the pattern of coordinate signs in the quadrants

the 1st quadrant = dn I	(+ +)
the 2nd quadrant = dn II	(- +)
the 3rd quadrant = dn III	(- -)
the 4th quadrant = dn IV	(+ -)

• starting with the quadrant (+ +)
number the quadrants
in a counterclockwise = positive direction
around the origin

Δ 3-space with a rectangular coordinate system
is separated by the 3 coordinate planes into
$2^3 = 8$ octants
according to coordinate signs
as pictured & described below:

• 3-space with a rectangular coordinate system
(visualize the rest)



• the 8 octants	the pattern of coordinate signs in the octants
the 1st octant = dn I	(+ + +)
the 2nd octant = dn II	(- + +)
the 3rd octant = dn III	(- - +)
the 4th octant = dn IV	(+ - +)
the 5th octant = dn V	(+ + -)
the 6th octant = dn VI	(- + -)
the 7th octant = dn VII	(- - -)
the 8th octant = dn VIII	(+ - -)

• starting with the octant (+ + +)
number the octants above the xy-plane
in the canonical direction,
then drop below the xy-plane
to the octant (+ + -)
which is below the octant (+ + +)
and continue to
number the octants below the xy-plane
in the canonical direction

Δ more generally for $2 \leq n \in$ pos int
to pass
from the canonical sequence of
the 2^n 2^n – ants of real n - space $\mathbb{R}^n$
to the canonical sequence of
2^{n+1} 2^{n+1} – ants of (n + 1) - space $\mathbb{R}^{n+1}$
write down twice
the pattern of coordinate signs for $\mathbb{R}^n$
& suffix plus $+$ to the first 2^n entries
& suffix minus $-$ to the last 2^n entries

it is a matter of judgement in a particular instance
as to the classification of
points with zero coordinates
ie
boundary points
of m-ants,
decreeing that a certain boundary point belongs to
none or one or many m-ants
according to a special purpose

Δ an m-ant
which includes all of its boundary points
may be called
closed
& designated with the use of an overbar as

$\overline{\mathrm{I}}$, $\overline{\mathrm{II}}$, $\overline{\mathrm{III}}$, etc

Δ an m-ant
which includes none of its boundary points
may be called
open
& designated with the use of an overcircle as

$\mathring{\mathrm{I}}$, $\mathring{\mathrm{II}}$, $\mathring{\mathrm{III}}$, etc

□ the chirality
of a 3-dimensioinal rectangular coordinate system
in physical 3-space

∆ the canonical correspondence between
the right/left hand
&
the coordinate system
= daf

• 1st form

pointing along
the positive

thumb .... x-axis

forefinger .... y-axis

midfinger .... z-axis

or

a cyclic permutation
of the coordinate axes
x-axis
y-axis
z-axis

• 2nd form

pointing along
the positive

thumb .... x-axis

forefinger & midfinger .... y-axis

ring & little fingers .... z-axis

or

a cyclic permutation
of the coordinate axes
x-axis
y-axis
z-axis

• 3rd form

pointing

curled fingers .... from the positive x-axis
to the positive y-axis

thumb .... along the positive z-axis

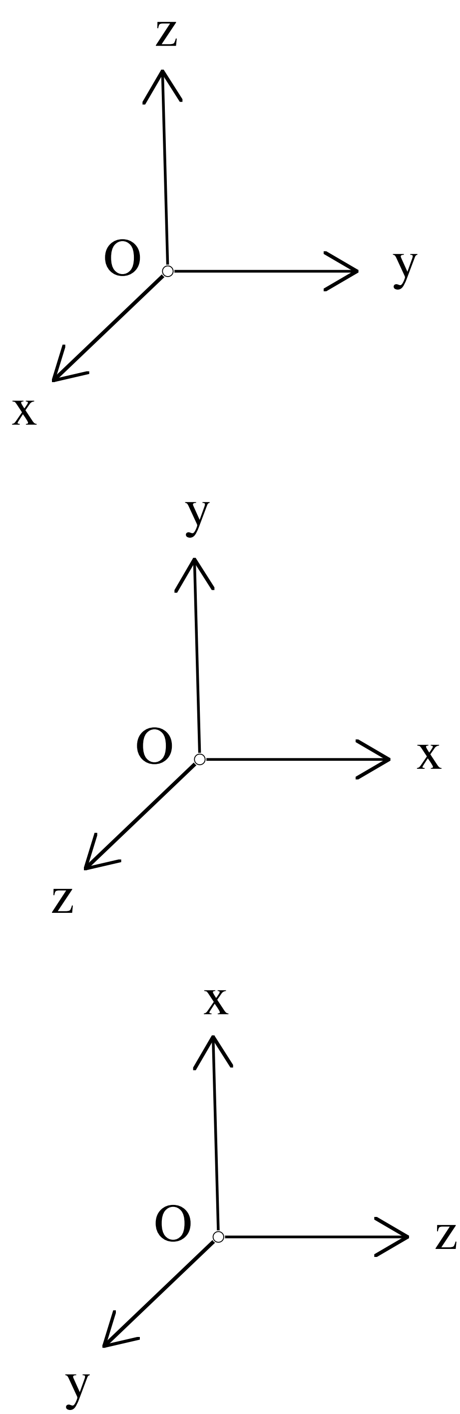
(note: one can think of the curled fingers
as wrapped around the z-axis
in the positive rotational direction)

or

a cyclic permutation
of the coordinate axes
x-axis
y-axis
z-axis

• the coordinate system
is
right-handed
or
left-handed
= df
the coordinare system
is in canonical correspondence with
the right hand
or
the left hand

Δ diagrams of
right-handed coordinate systems



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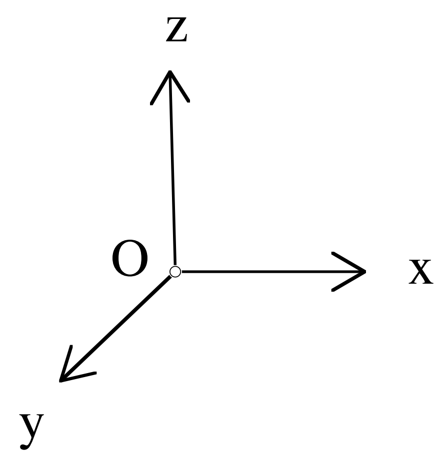
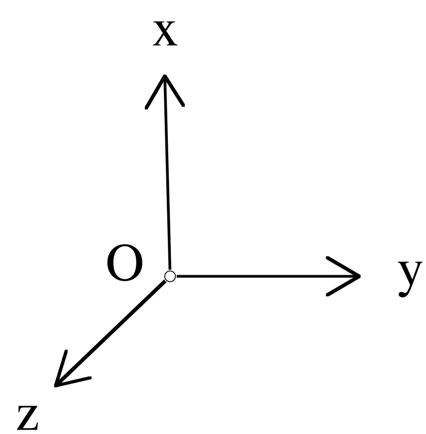
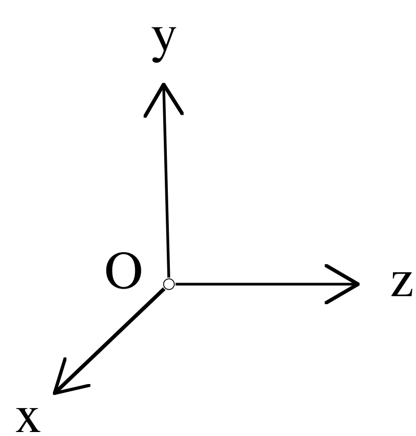
Δ diagrams of
left-handed coordinate systems







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Δ meanings of words

• chiral (adj)
= pr KI-ruhl
= df pertaining to the hand

• chirality (noun)
= pr ki-RAL-uh-tee
= df handedness

Δ etytmology

• chiral, chirality

come from

χειρ (Greek)
= hand

□ a theorem on elliptic quadrilaterals

T. for a quadrilateral inscribed in an ellipse
the two intersections
of the two pairs of opposite sides
&
the two intersections
of the two pairs of tangents at opposite vertices
are collinear

P. the proof consists in applying
a limiting case of Pascal's theorem twice

□ some geometric M's

∆ the two M's of plane geometry
are
• major
• minor
as in
the major/minor axis of an ellipse

∆ the three M's of solid geometry
are
• major
• mean
• minor
as in
the major/mean/minor axis of an ellipsoid

□ two complementary steps:
from algebra to geometry
&
from geometry to algebra

∆ the most critical two steps
in the history of mathematics
in recognizing the connection between
algebra & geometry
are described below
in present-day language
&
with the generous advantage of hindsight

∆ to repeat some standard algebraic definitions:
• the real number system
= a complete ordered field
• the cartesian plane
= the set of all ordered pairs of real numbers
• the pythagorean metric in the cartesian plane
= the distance function between two points
given by the formula in the pythagorean theorem
viz
the distance between two ordered number pairs
is the square root of the sum of the squares
of the differences between the coordinates

Δ from algebra to geometry;

by 1636 Fermat & in 1637 Descartes
made observations that lead to the statement:
the cartesian plane
equipped with the pythagorean metric
is a model of
euclidean plane geometry

Δ from geometry to algebra;

more than 260 years later in 1899
Hilbert proved the converse statement:
every model of euclidean plane geometry
is isomorphic to
the cartesian plane
equipped with the pythagorean metric

Δ the major part of Hilbert's achievement

was to find
a precise (albeit complicated) definition of
euclidean plane geometry
ie
to axiomatize euclidean plane geometry
completely & exactly
and thus
to finish the task begun
in Euclid's 'Elements' ca 300 BCE

□ curricula
from academic/scholarly environments
long long ago & far far away

∆ the ancient Greek Pythagoreans
regarded
Mathematics
as the study of two separate kinds of entities
viz

• The Discrete
= Numbers
meaning positive integers mostly
&
• The Continuous
= Magnitudes
in geometric objects
involving
lines & their lengths,
plane regions & their areas,
solids & their volumes

∆ according to Pythagorean doctrine
a mathematical entity can be
at rest
or
in motion;
whence

• Arithmetic
= the study of
The Discrete at Rest

• Music
= the study of
The Discrete in Motion

• Geometry
= the study of
The Continuous at Rest

• Astronomy
= the study of
The Continuous in Motion

∆ the Pythagorean doctrine of
The Quadrivium
which consists of the four subjects
Arithmetic
Music
Geometry
Astronomy
may be summarized by
The Quadrivium Tree
which is rooted in Mathematics
&
which has a double two-fold branching

Mathematics
↙ ↘
The Discrete = Numbers — The Continuous = Magnitudes
↓ ↓
At Rest — In Motion — At Rest — In Motion
↓ ↓ ↓ ↓
Arithmetic — Music — Geometry — Astronomy

∆ The Seven Liberal Arts

formed
the curriculum in medieval universities
which consisted of
the upper division

• The Quadrivium:
Arithmetic
Music
Geometry
Astronomy
which was the four-fold way to knowledge

plus
the lower division

• The Trivium:
Grammar
Rhetoric
Dialectics = Logic
which was the three-fold way to eloquence

Δ etymology

• quadrivium (Latin)
= meeting of four ways
= four-way crossroads
from
quadri- (Latin)
= four
+
via (Latin)
= road

• trivium (Latin)
= meeting of three ways
= three-way crossroads
from
tri- (Latin)
= three
+
via (Latin)
= road

Δ bionote

Martianus Capella,
a Latin writer of the 5th century CE
from northern Africa (probably Carthage),
originally conceived of
The Seven Liberal Arts
as the depository & summary of Roman culture
after the Fall of Rome
that (is usually said to have) occurred in 476 CE

□ the music of the spheres
is
an ancient Greek doctrine
that may have arisen in the following way

∆ Pythagoras observed that
strings in motion produce sounds
according to their lengths;
we now recognize that
a vibrating string's length
is inversely proportional to
its rate of motion = number of vibrations per second
& that determines its tone

• the heavenly bodies are in motion
& therefore produce sounds;
since all things in nature must harmonize,
the heavenly bodies produce
harmony/music
which, however, is too exquisite
to be heard by human ears

• each heavenly body
is understood to be fixed upon
a large invisible sphere centered at the Earth;
the heavenly bodies then move
because the ferrying spheres
carry them around the Earth

∆ thus is produced
the music of the spheres
where the word 'spheres' refers to
the ferrying spheres
or to
the heavenly bodies themselves

∆ Shakespeare described the music of the spheres
in his play
The Merchant of Venice
Act 5 Scene 1 lines 58-62
where the speaker Lorenzo is describing
a chart of the heavens

... Look how the flow of heaven
Is thick inlaid with patens of bright gold.
There's not the smallest orb which thou behold'st
But in his motion like an angel sings,
Still choiring to the young-eyed cherubins.

∆ bioline

Pythagoras of Samos
ca 580 - ca 500 BCE
Greek
geometer, philosopher,
founder of the Pythagorean Society

∆ bioline

William Shakespeare
1564 - 1616
English
dramatist & poet;
often considered to be
the greatest writer of all time

∆ note the presence of powers of 2

in Shakespeare's vital dates;
it is a good mnemonic