A Short Series of Serious Series Six-Packs

#56 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
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☐ we consider various power series
in the real variable x
with real coefficients;
let S be such a series;
we describe how to methodically alter S
in order to produce many power series
in the real variable x
with real coefficients
that are related to S;
this leads to the notions of
the three-pack of S,
the six-pack of S,
the triple six-pack of S,
etc
in a partial classification scheme
for power series;
this scheme also evidently applies to
other kinds of series
& to functons defined by series

• let
S = df a power series in the real variable x
with real coefficients
S = cl the admissible series

• the following notation & terminology is adopted:

ES = the series of the terms of S
containing the even powers of x
= the even-power admissible series

OS = the series of the terms of S
containing the odd powers of x
= the odd-power admissible series

AS = the series resulting from the change of sign
of every other term of S
= the alternating admissible series

DS = the tbt derivative of S wrt x
= the differentiated admissible series

IS = the tbt integral of S from 0 to x
= the integrated admissible series

• the three-pack of S
= df the following 3 series:

S = the admissible series
ES = the even-power admissible series
OS = the odd-power admissible series

• the six-pack of S
= df the following 6 series:

S = the admissible series
ES = the even-power admissible series
OS = the odd-power admissible series

AS = the alternating admissible series
AES = the alternating even-power admissible series
AOS = the alternating odd-power admissible series

• the triple six-pack of S
= df the following 18 series:

S = the admissible series
ES = the even-power admissible series
OS = the odd-power admissible series

AS = the alternating admissible series
AES = the alternating even-power admissible series
AOS = the alternating odd-power admissible series

DS = the differentiated admissible series
DES = the differentiated even-power admissible series
DOS = the differentiated odd-power admissible series

DAS = the differentiated alternating admissible series
DAES = the differentiated alternating even-power admissible series
DAOS = the differentiated alternating odd-power admissible series

IS = the integrated admissible series
IES = the integrated even-power admissible series
IOS = the integrated odd-power admissible series

IAS = the integrated alternating admissible series
IAES = the integrated alternating even-power admissible series
IAOS = the integrated alternating odd-power admissible series

□ the geometric power series
& its triple six - pack

• S

= the monic geometric power series with ratio x

$= 1 + x + x^2 + x^3 + \cdots$

$$= \sum_{n=0}^{\infty} x^n$$

$$= \frac{1}{1-x}$$

IC: $-1 < x < 1$

note: this series may be regarded as
the simplest nontrivial power series

• ES

$= \text{ the monic geometric power series with ratio } x^2$

$= 1 + x^2 + x^4 + x^6 + \cdots$

$$= \sum_{n=0}^{\infty} x^{2n}$$

$$= \frac{1}{1 - x^2}$$

IC: $-1 < x < 1$

• OS

$= \text{ the x - leading geometric power series with ratio } x^2$

$= x + x^3 + x^5 + x^7 + \cdots$

$$= \sum_{n=0}^{\infty} x^{2n+1}$$

$$= \frac{x}{1 - x^2}$$

IC: $-1 < x < 1$

• AS

= the alternating monic geometric power series in x

= the monic geometric power series with ratio $-x$

$$= 1 - x + x^2 - x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \frac{1}{1+x}$$

IC: $-1 < x < 1$

• AES

$= $ the monic geometric power series with ratio $-x^2$

$= 1 - x^2 + x^4 - x^6 + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= \frac{1}{1+x^2}$$

IC: $-1 < x < 1$

• AOS

$= \text{ the x - leading geometric power series with ratio } -x^2$

$= x - x^3 + x^5 - x^7 + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$$

$$= \frac{x}{1+x^2}$$

IC: $-1 < x < 1$

• DS

$= 1 + 2x + 3x^2 + 4x^3 \cdots$

$= \sum_{n=0}^{\infty} (n+1)\, x^n$

$= \dfrac{1}{(1-x)^2}$

IC: $-1 < x < 1$

• DES

$= 2x + 4x^3 + 6x^5 + 8x^7 + \cdots$

$= \sum_{n=0}^{\infty} 2(n+1)\, x^{2n+1}$

$= \dfrac{2x}{(1-x^2)^2}$

IC: $-1 < x < 1$

• DOS

$$= 1 + 3x^2 + 5x^4 + 7x^6 + \cdots$$

$$= \sum_{n=0}^{\infty} (2n+1)\, x^{2n}$$

$$= \frac{1 + x^2}{(1 - x^2)^2}$$

IC: $-1 < x < 1$

• $\overline{\text{D}}$AS

$$= 1 - 2x + 3x^2 - 4x^4 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

$$= \frac{1}{(1+x)^2}$$

IC: $-1 < x < 1$

note: the bar over the D
is suggestive of
a minus sign;
multiply the series DAS tbt with -1
to remove the initial minus sign

• $\overline{\text{D}}$AES

$= 2x - 4x^3 + 6x^5 - 8x^7 + \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \, 2(n+1) \, x^{2n+1}$

$= \dfrac{2x}{(1+x^2)^2}$

IC: $-1 < x < 1$

• DAOS

$$= 1 - 3x^2 + 5x^4 - 7x^6 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (2n+1) x^{2n}$$

$$= \frac{1 - x^2}{(1 + x^2)^2}$$

IC: $-1 < x < 1$

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• IS

$= $ the harmonic power series in x

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$= \log \frac{1}{1 - x}$$

IC: $-1 \leq x < 1$

• IES

$= $ the odd harmonic power series in x

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

$$= \log \sqrt{\frac{1+x}{1-x}}$$

$$= \tanh^{-1} x$$

IC: $-1 < x < 1$

• IOS

$=$ the even harmonic power series in x

$$= \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{2n}$$

$$= \log \frac{1}{\sqrt{1-x^2}}$$

IC: $-1 < x < 1$

• IAS

$= $ the alternating harmonic power series in x

$= $ Mercator' s series

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$= \log(1 + x)$$

IC: $-1 < x \le 1$

• IAES

= the alternating odd harmonic power series in x

= Gregory's series

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$= \tan^{-1} x$ (pv)

IC: $-1 < x \leq 1$

• IAOS

= the alternating even harmonic power series in x

$$= \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n}$$

$$= \log \sqrt{1 + x^2}$$

IC: $-1 \le x \le 1$

☐ the harmonic power series
& its six-pack

• S

= the harmonic power series in x

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$= \log \frac{1}{1 - x}$$

IC: $-1 \le x < 1$

• OS

= the odd harmonic power series in x

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

$$= \log \sqrt{\frac{1+x}{1-x}}$$

$$= \tanh^{-1} x$$

IC: $-1 < x < 1$

• ES

$= $ the even harmonic power series in x

$$= \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{2n}$$

$$= \log \frac{1}{\sqrt{1 - x^2}}$$

IC: $-1 < x < 1$

• AS

= the alternating harmonic power series in x

= Mercator's series

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$= \log(1 + x)$$

IC: $-1 < x \le 1$

• AOS

$= $ the alternating odd harmonic power series in x

$= $ Gregory' s series

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= \tan^{-1} x \ \ (\text{pv})$$

IC: $-1 < x \leq 1$

• AES

$= $ the alternating even harmonic power series in x

$$= \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n}$$

$$= \log \sqrt{1 + x^2}$$

IC: $-1 \le x \le 1$

□ the factorial power series
& its six - pack

• S

= the factorial power series in x

= the exponential series in x

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= e^x$$

IC: $-\infty < x < \infty$

• ES

= the even factorial power series in x

= the hyperbolic cosine series in x

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$= \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

IC: $-\infty < x < \infty$

• OS

$= $ the odd factorial power series in x

$= $ the hyperbolic sine series in x

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$= \frac{e^x - e^{-x}}{2}$$

$$= \sinh x$$

IC: $-\infty < x < \infty$

• AS

= the alternating factorial power series in x

= the alternating exponential series in x

$$= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$= e^{-x}$$

IC: $-\infty < x < \infty$

• AES

$= $ the alternating even factorial power series in x

$= $ the cosine series in x

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= \frac{e^{ix} + e^{-ix}}{2}$$

$$= \cos x$$

IC: $-\infty < x < \infty$

• AOS

= the alternating odd factorial power series in x

= the sine series in x

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \frac{e^{ix} - e^{-ix}}{2i}$$

$$= \sin x$$

IC: $-\infty < x < \infty$

☐ every real function f(x)
with a power series expansion in x
say

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

with a plural IC
can be decomposed into
the sum of
an even function

$$g(x) = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots = \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

and
an odd function

$$h(x) = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \cdots = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

whence

$f(x) = g(x) + h(x)$

&

$g(x) = \frac{1}{2}[f(x) + f(-x)]$

is called the even part of f(x)

&

$h(x) = \frac{1}{2}[f(x) - f(-x)]$

is called the odd part of f(x)

&

the three – pack of f(x)

consists of

$f(x),\ g(x),\ h(x)$

note: the above equations evidently hold

even for a function without a series development if we take

$S = f(x)$

then

$ES = g(x)$

$OS = h(x)$

$AS = f(-x)$

$AES = g(i\,x)$

$AOS = \frac{1}{i} h(ix)$

we illustrate the even - odd decomposition
of a function and its series implications
by two examples,
one in the real field & one in the complex field

the real exponential function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

has

the hyperbolic cosine

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

as even part

&

the hyperbolic sine

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

as odd part

&

the even - odd decomposition

$$e^x = \cosh x + \sinh x$$

(Lambert' s formula)

the complex - valued exponential function

$$e^{ix} = 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \cdots$$

has

the trig cosine

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

as even part

&

the trig sine times i

$$i \sin x = i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$$

as odd part

&

the even - odd decomposition

$$e^{ix} = \cos x + i \sin x$$

(Euler' s formula)

□ the above even - odd decompositions also hold
for the corresponding complex functions
viz

$e^{z} = \cosh z + \sinh z$ (Lambert's formula)

$e^{iz} = \cos z + i \sin z$ (Euler's formula)

wh $z \in$ complex var

the functions being defined by
the replacement of x by z
in the real series developments

□ the considerations
leading to the the notion of
the six-pack
of a real power series in x
or
of a function so defined
may be extended to
a series which has any functions
as terms of the series
or
of a function so defined;
here the attention is directed to
the number of the term in the series
rather than the exponent of x;
an example starting with the zeta function
is given below

□ the zeta function
& its six-pack

• the zeta function of Riemann
= the ζ-function of Riemann
= the Riemann zeta function
= the Riemann ζ-function
= Riemann's zeta function
= Riemann's ζ-function
= the zeta function
= the ζ-function

$=_{dn}$ $\zeta(x)$

$=_{rd}$ zeta (of) x

$$=_{df} \; 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots$$

$$= \; 1 + 2^{-x} + 3^{-x} + 4^{-x} + \cdots$$

$$= \; \sum_{n=1}^{\infty} \frac{1}{n^x}$$

$$= \; \sum_{n=1}^{\infty} n^{-x}$$

IC: $1 < x < \infty$

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• the lambda function

$= \text{ the } \lambda - \text{function}$

$=_{dn} \ \lambda(x)$

$=_{rd}$ lambda (of) x

$$=_{df} \ 1 + \frac{1}{3^x} + \frac{1}{5^x} + \frac{1}{7^x} + \cdots$$

$$= \ 1 + 3^{-x} + 5^{-x} + 7^{-x} + \cdots$$

$$= \ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x}$$

$$= \ \sum_{n=0}^{\infty} (2n+1)^{-x}$$

IC: $1 < x < \infty$

• the kappa function

$= \text{ the } \kappa - \text{function}$

$=_{dn} \ \kappa(x)$

$=_{rd}$ kappa (of) x

$$=_{df} \ \frac{1}{2^x} + \frac{1}{4^x} + \frac{1}{6^x} + \frac{1}{8^x} + \cdots$$

$$= \ 2^{-x} + 4^{-x} + 6^{-x} + 8^{-x} + \cdots$$

$$= \ \sum_{n=1}^{\infty} \frac{1}{(2n)^x}$$

$$= \ \sum_{n=1}^{\infty} (2n)^{-x}$$

IC: $1 < x < \infty$

• the eta function

= the η - function

$=_{dn}$ $\eta(x)$

$=_{rd}$ eta (of) x

$$=_{df} \ 1-\frac{1}{2^x}+\frac{1}{3^x}-\frac{1}{4^x}+\cdots$$

$$= \ 1-2^{-x}+3^{-x}-4^{-x}+\cdots$$

$$= \ \sum_{n=1}^{\infty}(-1)^{n+1}\ \frac{1}{n^x}$$

$$= \ \sum_{n=1}^{\infty}(-1)^{n+1}\ n^{-x}$$

IC: $0 < x < \infty$

- the beta function

$= $ the $\beta -$ function

$=_{dn}$ $\beta(x)$

$=_{rd}$ beta (of) x

$$=_{df} \; 1 - \frac{1}{3^x} + \frac{1}{5^x} - \frac{1}{7^x} + \cdots$$

$$= \; 1 - 3^{-x} + 5^{-x} - 7^{-x} + \cdots$$

$$= \; \sum_{n=0}^{\infty} (-1)^n \; \frac{1}{(2n+1)^x}$$

$$= \; \sum_{n=0}^{\infty} (-1)^n \; (2n+1)^{-x}$$

IC: $0 < x < \infty$

• the alpha function

$= \text{ the } \alpha - \text{function}$

$=_{dn} \ \alpha(x)$

$=_{rd}$ alpha (of) x

$$=_{df} \ \frac{1}{2^x} - \frac{1}{4^x} + \frac{1}{6^x} - \frac{1}{8^x} + \cdots$$

$$= 2^{-x} - 4^{-x} + 6^{-x} - 8^{-x} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n)^x}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} (2n)^{-x}$$

IC: $0 < x < \infty$

□ in summary

the zeta function & its six - pack

- $\zeta(x) = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots$

- $\lambda(x) = 1 + \frac{1}{3^x} + \frac{1}{5^x} + \frac{1}{7^x} + \cdots$

- $\kappa(x) = \frac{1}{2^x} + \frac{1}{4^x} + \frac{1}{6^x} + \frac{1}{8^x} + \cdots$

- $\eta(x) = 1 - \frac{1}{2^x} + \frac{1}{3^x} - \frac{1}{4^x} + \cdots$

- $\beta(x) = 1 - \frac{1}{3^x} + \frac{1}{5^x} - \frac{1}{7^x} + \cdots$

- $\alpha(x) = \frac{1}{2^x} - \frac{1}{4^x} + \frac{1}{6^x} - \frac{1}{8^x} + \cdots$

note that:

$$\zeta(x) = \lambda(x) + \kappa(x) = \eta(x) + 2\kappa(x)$$

$$\eta(x) = \lambda(x) - \kappa(x) = \zeta(x) - 2\kappa(x)$$

$$\zeta(x) = 2^x \kappa(x)$$

$$\eta(x) = 2^x \alpha(x)$$