Adjoining Infinities to Number Systems

#55 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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GG55-2

□ the real projective line

• the one - element extension of the real number system $\mathbb{R}$
= the one - point compactification of the real number line $\mathbb{R}$
= the real projective line
= $\mathbb{R} \cup \{\infty\}$
$=_{dn}$ $\dot{\mathbb{R}}$
$=_{rd}$ (open cap) ar (overscript) dot
wh
∞
$=_{rd}$ infinity
$=_{cl}$ the real projective point at infinity

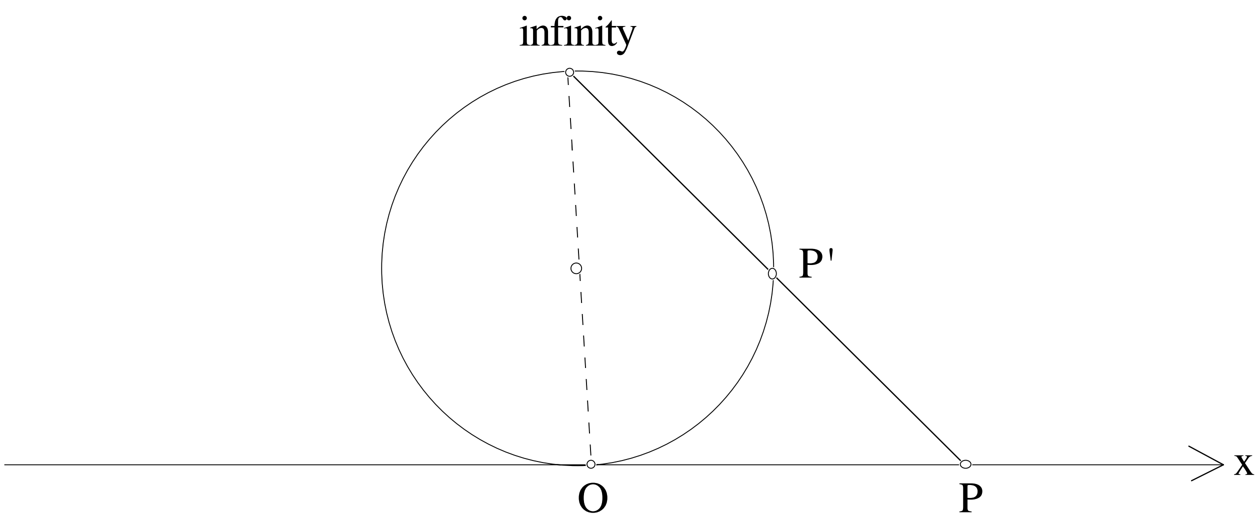
note that the dot above $\mathbb{R}$ in $\dot{\mathbb{R}}$
suggests the adjoined point at infinity

• there is nothing mysterious about ∞;
it is only necessary to choose ∞
as a set that is not an element of $\mathbb{R}$;
the choice $\infty =_{df} \mathbb{R}$ is satisfactory
since no set is an element of itself
by the axiom of foundation

• to make $\dot{\mathbb{R}}$ into a topological space,
define neighborhoods of points of $\dot{\mathbb{R}}$ as follows:
for a point of $\mathbb{R}$
a neighborhood is any subset of $\dot{\mathbb{R}}$
that contains an open interval of $\mathbb{R}$
that contains the point;
for ∞
a neighborhood is any subset of $\dot{\mathbb{R}}$
that contains ∞
and both a left ray & a right ray of $\mathbb{R}$;
this makes the real projective line $\dot{\mathbb{R}}$
into a circle topologically;
a geometric construction to show this
is given below

• GP

polar projection
relating line & circle



GG55-5

• since $\dot{\mathbb{R}}$ is a simple closed curve,
the linear order in $\mathbb{R}$
cannot be meaningfully extended to $\dot{\mathbb{R}}$;

• some algebraic operations in $\mathbb{R}$
are extended to $\dot{\mathbb{R}}$ as follows
wh r is any real number:

$-\infty = \infty$

$\infty + \infty = \infty$

$\infty \times \infty = \infty$

$r + \infty = \infty + r = \infty$

$r \times \infty = \infty \times r = \infty$ if $r \neq 0$

$r / \infty = 0$

$\infty / r = \infty$

$r / 0 = \infty$ if $r \neq 0$

□ the extended real line

• the two - element extension of the real number system $\mathbb{R}$
= the two - point compactification of the real number line $\mathbb{R}$
= the bilaterally extended real number system / line
= the extended real line
= $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$
$=_{dn}$ $\overline{\mathbb{R}}$
$=_{rd}$ (open cap) ar (overscript) bar
wh
$-\infty$
$=_{rd}$ minus infinity
$=_{cl}$ the negative real point at infinity
&
$+\infty$
$=_{rd}$ plus infinity
$=_{cl}$ the positive real point at infinity

note that the bar above $\mathbb{R}$ in $\overline{\mathbb{R}}$
suggests the topological closure of $\mathbb{R}$

GG55-7

• conveniently

$+\infty =_{\text{df}} (\mathbb{R}, 1)$

$-\infty =_{\text{df}} (\mathbb{R}, -1)$

as distinct sets that are not elements of $\mathbb{R}$

• evidently

the real number line $\mathbb{R}$

can be extended by one infinity at a time

to

$\mathbb{R} \cup \{+\infty\}$

or

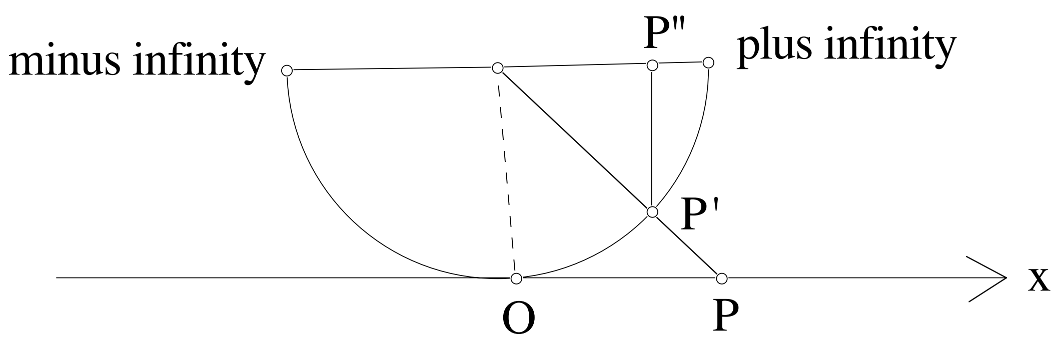
to

$\{-\infty\} \cup \mathbb{R}$

• to make $\overline{\mathbb{R}}$ into a topological space,
define neighborhoods of points of $\overline{\mathbb{R}}$ as follows:
for a point of $\mathbb{R}$
a neighborhood is any subset of $\overline{\mathbb{R}}$
that contains an open interval of $\mathbb{R}$
that contains the point;
for $+\infty$
a neighborhood is any subset of $\overline{\mathbb{R}}$
that contains $+\infty$
and a right ray of $\mathbb{R}$;
for $-\infty$
a neighborhood is any subset of $\overline{\mathbb{R}}$
that contains $-\infty$
and a left ray of $\mathbb{R}$;
this makes the extended real line $\overline{\mathbb{R}}$
into a closed line segment topologically;
a geometric construction to illustrate this
is given below

• GP

central projection
relating line & semicircle & diameter



- to extend the linear order in $\mathbb{R}$

to a linear order in $\overline{\mathbb{R}}$:

define

$-\infty < +\infty$

$-\infty < r < +\infty$

wh r is any real number

- some algebraic operations in $\mathbb{R}$

are extended to $\overline{\mathbb{R}}$ as follows

wh r is any real number:

$-(+\infty) \;=\; -\infty$

$-(-\infty) \;=\; +\infty$

$(+\infty) + (+\infty) \;=\; +\infty$

$(-\infty) + (-\infty) \;=\; -\infty$

$(+\infty) - (-\infty) \;=\; +\infty$

$(-\infty) - (+\infty) \;=\; -\infty$

$(+\infty) \times (+\infty) \;=\; +\infty$

$(-\infty) \times (-\infty) \;=\; +\infty$

$(+\infty) \times (-\infty) \;=\; (-\infty) \times (+\infty) \;=\; -\infty$

$$r + (+\infty) = (+\infty) + r = +\infty$$
$$r + (-\infty) = (-\infty) + r = -\infty$$
$$r - (+\infty) = -\infty$$
$$r - (-\infty) = +\infty$$
$$(+\infty) - r = +\infty$$
$$(-\infty) - r = -\infty$$
$$r \times (+\infty) = (+\infty) \times r = +\infty \text{ if } r > 0$$
$$r \times (+\infty) = (+\infty) \times r = -\infty \text{ if } r < 0$$
$$r \times (-\infty) = (-\infty) \times r = -\infty \text{ if } r > 0$$
$$r \times (-\infty) = (-\infty) \times r = +\infty \text{ if } r < 0$$
$$r / (+\infty) = r / (-\infty) = 0$$
$$(+\infty) / r = +\infty \text{ if } r > 0$$
$$(+\infty) / r = -\infty \text{ if } r < 0$$
$$(-\infty) / r = -\infty \text{ if } r > 0$$
$$(-\infty) / r = +\infty \text{ if } r < 0$$

□ the complex sphere

• the one - element extension
of the complex number system $\mathbb{C}$
= the one - point compactification
of the complex number plane $\mathbb{C}$
= the extended complex number plane
= the extended complex plane
= the complex number sphere
= the complex sphere
= the Riemann sphere
= $\mathbb{C} \cup \{\infty\}$
$=_{dn}$ $\dot{\mathbb{C}}$
$=_{rd}$ (open cap) cee (overscript) dot
wh
∞
$=_{rd}$ infinity
$=_{cl}$ the complex point at infinity

note that the dot above $\mathbb{C}$ in $\dot{\mathbb{C}}$
suggests the adjoined point at infinity

• conveniently

$\infty =_{df} \mathbb{C}$

since $\mathbb{C}$ is not an element of $\mathbb{C}$

• to make $\dot{\mathbb{C}}$ into a topological space,

define neighborhoods of points of $\dot{\mathbb{C}}$ as follows:

for a point of $\mathbb{C}$

a neighborhood is any subset of $\dot{\mathbb{C}}$

that contains an open disc of $\mathbb{C}$

that contains the point;

for ∞

a neighborhood is any subset of $\dot{\mathbb{C}}$

that contains ∞

and the complement in $\mathbb{C}$

of any bounded subset of $\mathbb{C}$;

tbis makes the complex sphere $\dot{\mathbb{C}}$

into (the surface of) a sphere in 3 - space topologically;

a geometric construction to illustrate this

is given below

• GP

stereographic polar projection
relating plane & sphere

visualize by the mind's eye:
a sphere is tangent to the horizontal
xy-plane = z-plane
from above at the origin
which is then also the south pole of the sphere;
draw a line segment from the north pole of the sphere
to a point P of the plane;
this line segment intersects the sphere in a point P′;
as the point P ranges over the plane,
the point P′ ranges over the punctured sphere
ie the sphere without the north pole;
the north pole represents
the complex point at infinity

• some algebraic operations in $\mathbb{C}$ are extended to $\dot{\mathbb{C}}$ as follows wh z is a complex number:

$$-\infty = \infty$$

$$\infty + \infty = \infty$$

$$\infty \times \infty = \infty$$

$$z + \infty = \infty + z = \infty$$

$$z \times \infty = \infty \times z = \infty \text{ if } z \neq 0$$

$$z / \infty = 0$$

$$\infty / z = \infty$$

$$z / 0 = \infty \text{ if } z \neq 0$$