Math Chants

#53 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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500 Angell St #414
Providence RI 02906

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□ mathematical chants

• sometimes reading/saying out loud
a formula or a theorem
or a comment or a paraphrase
or a mnemonic or etc
becomes a kind of chant
which may help
the understanding
&
the memory;
call it
symbols into speech;
call it
mathematical metrical speaking;
call it
rhythmic math;
call it
rock & roll math;
call it
math chant;
call it
math rap;
here are some examples

□ multiplicative sign rule

$(a)(b) = ab$

$(a)(-b) = -ab$

$(-a)(b) = -ab$

$(-a)(-b) = ab$

• one stays & two disappear;

plus times plus gives plus

plus times minus gives minus

minus times plus gives minus

minus times minus gives plus

• one minus gives minus

two minuses give plus

• in a continued product

an even number of minuses gives a plus

an odd number of minuses gives a minus

☐ the order of arithmetic/algebraic operations is described by the initial-letter mnemonic

• Please Excuse My Dear Aunt Sally
= PEMDAS

where
P = parentheses
E = exponents
M = multiply
D = divide
A = add
S = subtract

☐ the two - binomial product expansion rule

$(a+b)(c+d) = ac + ad + bc + bd$

• for mnemonic

say: foil

where the letters are initials of the phrases

f = first terms

o = outside terms

i = inside terms

l = last terms

multiply & add

☐ the quadratic formula

$a x^2 + b x + c = 0$

$\Leftrightarrow$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

wh

$a, b, c \in$ complex nr

& $a \neq 0$

& $x \in$ complex nr var

• ex equals
minus bee
plus or minus
the square root of
bee square minus four ay cee
all divided by
two ay

☐ the product & quotient rules for the logarithm

$$\log AB = \log A + \log B$$

$$\log \frac{A}{B} = \log A - \log B$$

☐ the power & root rules for the logarithm

$$\log A^n = n \log A$$

$$\log \sqrt[n]{A} = \frac{\log A}{n}$$

• the logarithm converts
multiplication & division
into
addition & subtraction:
the log of the product of two positive numbers
equals
the sum of the logs of the numbers
&
the log of the quotient of two positive numbers
equals
the difference of the logs of the numbers

• log of product
equals
sum of logs
&
log of quotient
equals
difference of logs

• the logarithm converts
power raising & root extraction
into
multiplication & division:
the log of a power
equals
the exponent times the log of the base of the power
&
the log of a root
equals
the log of the radicand divided by the index of the root

• log of power
equals
expo times log of base
&
log of root
equals
log of rad over index

☐ the isosceles triangle theorem

• two sides of a triangle are equal
if and only if
the opposite angles are equal

☐ the pythagorean theorem

• the square of the hypotenuse of a right triangle
equals
the sum of the squares of the two legs

• more succinctly:
square of hypo of right triangle
equals
sum of squares of legs

☐ the concurrency theorem

• for any triangle
these four triplets of lines are each concurrent:
the altitudes,
the medians,
the internal angle bisectors,
the side perpendicular-bisectors

□ area K of triangles & quads ito base b & height h

• area of triangle

$$K = \frac{1}{2} b h$$

area of triangle

equals

one - half base times height

• area of rectangle & parallelogram

$$K = b h$$

area of rectangle & parallelogram

equals

base times height

• area of trapezoid

$$K = \frac{1}{2}(b_1 + b_2) h$$

area of trapezoid

equals

arithmetic mean of bases times height

□ area K of a triangle ito sides & angles

• ito one side & three angles

$$K = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{b^2 \sin C \sin A}{2 \sin B} = \frac{c^2 \sin A \sin B}{2 \sin C}$$

• ito two sides & the included angle

$$K = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C$$

• ito three sides

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

• the above formulas may best be read as written;
the middle formulas have the paraphrase:
the area of a triangle
equals
one-half the product of any two sides
times
the sine of the included angle

□ the circle formulas

$$C = \pi d$$

• circumference of circle
equals
pi times diameter

$$C = 2\pi r$$

• circumference of circle
equals
two pi times radius

$$A = \frac{1}{4}\pi d^2$$

• area of circle
equals
one-fourth pi times diameter squared

$$A = \pi r^2$$

• area of circle
equals
pi times radius squared

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☐ trig fcns of an acute angle of a right triangle

ito the sides

$$\sin = \frac{opp}{hyp}$$

$$\cos = \frac{adj}{hyp}$$

$$\tan = \frac{opp}{adj}$$

the initial letters spell the name of

The Great Chief

SOHCAHTOA

☐ the addition formula for the sine

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

• the sine of the sum of two angles
equals
the sine of the first times the cosine of the second
plus
the cosine of the first times the sine of the second

• sine of sum
equals
sine of first times cosine of second
plus
cosine of first times sine of second

☐ the subtraction formula for the sine

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

• the sine of the difference of two angles
equals
the sine of the first times the cosine of the second
minus
the cosine of the first times the sine of the second

• sine of difference
equals
sine of first times cosine of second
minus
cosine of first times sine of second

□ the addition formula for the cosine

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

• the cosine of the sum of two angles

equals

the product of the cosines

minus

the product of the sines

• cosine of sum

equals

product of cosines

minus

product of sines

☐ the subtraction formula for the cosine

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

• the cosine of the difference of two angles
equals
the product of the cosines
plus
the product of the sines

• cosine of difference
equals
product of cosines
plus
product of sines

☐ the addition formula for the tangent

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

• the tangent of the sum of two angles
equals
the sum of the tangents
over
one minus the product of the tangents

• tangent of sum
equals
sum of tangents
over
one minus product of tangents

☐ the subtraction formula for the tangent

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

• the tangent of the difference of two angles
equals
the difference of the tangents
over
one plus the product of the tangents

• tangent of difference
equals
difference of tangents
over
one plus product of tangents

☐ the law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

• for any triangle
the ratio of any side to the sine of the opposite angle
equals
the circumdiameter

• for any triangle
the ratio of the sides
equals
the ratio of the sines of the opposite angles
(rather than the ratio of the angles themselves)

□ the law of cosines

$$a^2 = b^2 + c^2 - 2\,b\,c\cos A$$

$$b^2 = c^2 + a^2 - 2\,c\,a\cos B$$

$$c^2 = a^2 + b^2 - 2\,a\,b\cos C$$

• for any triangle
the square of any side
equals
the sum of the squares of the other two sides
minus
twice their product
times
the cosine of the included angle

• the law of cosines
generalizes
the pythagorean theorem

☐ slope formula

slope m of a straight line in the plane

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

• slope $= \frac{\text{rise}}{\text{run}}$

• slope equals rise over run

☐ distance formula

distance d between two points in the plane

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

• distance $= \sqrt{(\text{run})^2 + (\text{rise})^2}$

• distance

equals

square root of run squared plus rise squared

□ note on notation

the letter d for distance
is evidently suggested by the initial letter
of the word ‘distance’;
¿ but why is m the standard letter to denote slope ?
it has been suggested that the letter m for slope
comes from the initial letter of the French verb
monter = to climb;
the thought is ingenious
and indeed
both independent discoverers,
Descartes & Fermat,
of analytic geometry were French;
however, I conjecture that the suggestion
is more poetry than history;
perhaps the idea can be regarded as a mnemonic

□ eccentricity e of conic sections

- circle: $e = 0$
- ellipse: $0 < e < 1$
- parabola: $e = 1$
- hyperbola: $e > 1$
- equilateral / rectangular hyperbola: $e = \sqrt{2}$

- e measures shape:

all circles are similar

all parabolas are similar

all equilateral / rectangular hyperbolas are similar

☐ the sum formula for the derivative

$$(u+v)' = u' + v'$$

• the derivative of the sum of two functions
equals
the sum of the derivatives

• derivative of sum
equals
sum of derivatives

□ the difference formula for the derivative

$(u - v)' = u' - v'$

• the derivative of the difference of two functions
equals
the difference of the derivatives

• derivative of difference
equals
difference of derivatives

□ the product formula for the derivative

$(u\,v)' = u\,v' + v\,u'$

- the derivative of the product of two functions
equals
the first times the derivative of the second
plus
the second times the derivative of the first

- derivative of product
equals
first times derivative of second
plus
second times derivative of first

☐ the quotient formula for the derivative

$$\left(\frac{u}{v}\right)' = \frac{v\,u' - u\,v'}{v^2}$$

• the derivative of the quotient of two functions
equals
the denominator times the derivative of the numerator
minus
the numerator times the derivative of the denominator
all divided by
the square of the denominator

• derivative of quotient
equals
denominator times derivative of numerator
minus
numerator times derivative of denominator
all over
square of denominator

□ also note

$$\left(\frac{u}{v}\right)' = \frac{1}{v^2}\begin{vmatrix} u' & v' \\ u & v \end{vmatrix} = \begin{vmatrix} \frac{u'}{v} & \frac{v'}{v} \\ \frac{u}{v} & \frac{v}{v} \end{vmatrix}$$

easy to look at
& remember
but
hard to say

☐ the power rule for the derivative

$$\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}$$

• the derivative of a power of a function
equals
the exponent
times
the base to a power one less
times
the derivative of the base

• derivative of power
equals
exponent
times
base to power one less
times
derivative of base

☐ the exponential rule for the derivative

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$\frac{d}{dx} e^x = e^x$$

• the derivative of the exponential of a function
equals
the exponential of the function
times
the derivative of the function

• derivative of expo fcn is itself

☐ the logarithm rule for the derivative

$$\frac{d}{dx}\log u = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}\log x = \frac{1}{x}$$

• the derivative of the logarithm of a function
equals
the reciprocal of the function
times
the derivative of the function

• derivative of log fcn is recip fcn

☐ the logarithm absolute rule for the derivative

$$\frac{d}{dx} \log|u| = \frac{1}{u} \frac{du}{dx}$$

$$\frac{d}{dx} \log|x| = \frac{1}{x}$$

• the derivative of the logarithm absolute of a function
equals
the reciprocal of the function
times
the derivative of the function

• derivative of log abso fcn is recip fcn

☐ the continued product rule for the derivative

• to differentiate a continued product
differentiate each factor separately
and add

☐ to take the nth derivative of a product uv,
use the binomial theorem to expand the RHS
& replace each rth power by the rth derivative:

$$D^n (uv) = (D\,u + D\,v)^n$$

such notational prestidigitation
of suddenly changing the notation
is called
the umbral calculus

☐ derivatives of the 6 trig fcns

$$D = \frac{d}{dx}$$

- $D \sin x = \cos x$
- $D \cos x = -\sin x$
- $D \tan x = \sec^2 x$
- $D \cot x = -\csc^2 x$
- $D \sec x = \sec x \tan x$
- $D \csc x = -\csc x \cot x$

note:
interchange of cofcns on LHS
produces
interchange of cofcns on RHS
together with change of sign

- derivative of sine is cosine
- derivative of cosine is minus sine
- derivative of tangent is secant squared
- derivative of cotangent is minus cosecant squared
- derivative of secant is secant times tangent
- derivative of cosecant is minus cosecant times cotangent

□ the linear rule for the integral

$$\int (\alpha u + \beta v)\,dx = \alpha \int u\,dx + \beta \int v\,dx$$

$$\int_a^b (\alpha u + \beta v)\,dx = \alpha \int_a^b u\,dx + \beta \int_a^b v\,dx$$

• for indefinite & definite integrals

the integral is a linear function of the integrand

☐ the tracking rule for the definite integral

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

• for the definite integral

integral from a to b plus integral from b to c

equals

integral from a to c

☐ the backtracking rule for the definite integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

• for the definite integral

interchanging lims changes sign

□ the formula for integration by parts

$$\int u\,dv = u\,v - \int v\,du$$

• integral of yu dee vee
equals
yu times vee
minus
integral of vee dee yu

☐ Green's Formula / Theorem

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} dx\,dy$$

looking is better than reading

☐ Stokes' Formula / Law / Theorem

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

looking is better than reading;
it's a very pretty & a very powerful formula

☐ De Morgan laws for conjunction & disjunction

$\neg (p \,\&\, q) \Leftrightarrow \neg p \vee \neg q$

$\neg (p \,\&\, q \,\&\, r) \Leftrightarrow \neg p \vee \neg q \vee \neg r$

etc

$\neg (p \vee q) \Leftrightarrow \neg p \,\&\, \neg q$

$\neg (p \vee q \vee r) \Leftrightarrow \neg p \,\&\, \neg q \,\&\, \neg r$

etc

• the negation of a conjunction of propositions
is equivalent to
the disjunction of the negations of the propositions
& dually
the negation of a disjunction of propositions
is equivalent to
the conjunction of the negations of the propositions

• negation of conjunction
is equivalent to
disjunction of negations
& dually
negation of disjunction
is equivalent to
conjunction of negations

• not and iff or nots
& dually
not or iff and nots

☐ De Morgan laws for quantifiers

$\neg\forall x.fx \Leftrightarrow \exists x.\neg fx$

$\neg\exists x.fx \Leftrightarrow \forall x.\neg fx$

• it is not the case that for all x, fx
if and only if
there exists x such that it is not the case that fx
& dually
it is not the case that there exists x such fx
if and only if
for all x, it is not the case that fx

• not all is equivalent to some not
& dually
not some is equivalent to all not

• not all ef is some not ef
& dually
not some ef is all not ef

□ De Morgan laws for intersection & union of sets

$(A \cap B)' = A' \cup B'$

$(A \cap B \cap C)' = A' \cup B' \cup C'$

etc

$(A \cup B)' = A' \cap B'$

$(A \cup B \cup C)' = A' \cap B' \cap C'$

etc

$$\left(\bigcap_{i \in I} A_i\right)' = \bigcup_{i \in I} A_i'$$

$$\left(\bigcup_{i \in I} A_i\right)' = \bigcap_{i \in I} A_i'$$

• complement of intersection
equals
union of complements
& dually
complement of union
equals
intersection of complements

□ here is a famous example
of presumably unconscious versification
with perfect meter & rhyme
in the midst of a serious mathematical treatise:

• And so no force however great
can stretch a cord however fine
into an horizontal line
which is accurately straight.
(the last line is alternatively quoted as
'that shall be absolutely straight.')

from
'Elementary Treatise on Mechanics' (1819)
by
William Whewell
1794-1866
English
mathematician, scientist,
philosopher, literary scholar