Three Momentous Means: AM, GM, AGM

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GG52-2

D. the arithmetic mean

= AM

let

- $a, b \in$ real nr

then

- the arithmetic mean of a and b

$=_{dn}$ AM(a, b)

$=_{df} \dfrac{a+b}{2}$

note AM $\leftarrow$ the capitalized initial letters of arithmetic mean

D. the geometric mean

= GM

let

- $a, b \in$ real nr
- $a, b \geq 0$

then

- the geometric mean of a and b

$=_{dn}$ GM(a, b)

$=_{df}$ $\sqrt{a\,b} \geq 0$

note GM $\leftarrow$ the capitalized initial letters of geometric mean

T. the arithmetic mean & geometric mean inequality
= the AM & GM inequality

let

- $a, b \in$ real nr
- $a, b \geq 0$

then

- $\max(a, b) \geq AM(a, b) \geq GM(a, b) \geq \min(a, b)$
- $AM(a, b) > GM(a, b) \iff a \neq b$
- $AM(a, b) = GM(a, b) \iff a = b$

- $a \leq b \iff a \leq GM(a, b) \leq AM(a, b) \leq b$
- $a < b \iff a \leq GM(a, b) < AM(a, b) < b$
- $0 < a < b \iff a < GM(a, b) < AM(a, b) < b$
- $0 = a < b \iff a = GM(a, b) < AM(a, b) < b$
- $a = b \iff a = GM(a, b) = AM(a, b) = b$

- $GM(a, b) = a \iff a = 0 \vee a = b$
- $GM(a, b) = b \iff b = 0 \vee a = b$
- $AM(a, b) = a \iff a = b$
- $AM(a, b) = b \iff a = b$

P. in two parts

(1) $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + 2ab + b^2 \geq 4ab$

$(a+b)^2 \geq 4ab$

$a + b \geq 2\sqrt{ab}$

$\frac{a+b}{2} \geq \sqrt{ab}$

$AM(a,b) \geq GM(a,b)$

(2) tfsape

$a = b$

$a - b = 0$

$(a - b)^2 = 0$

$a^2 - 2ab + b^2 = 0$

$a^2 + 2ab + b^2 = 4ab$

$(a + b)^2 = 4ab$

$a + b = 2\sqrt{ab}$

$$\frac{a + b}{2} = \sqrt{ab}$$

$AM(a, b) = GM(a, b)$

the proof is readily completed

R. let

- $a, b \in$ real nr
- $a, b \geq 0$

then

- $0 \leq AM(a,b) - GM(a,b) \leq \frac{1}{2}|a-b|$
- $0 < AM(a,b) - GM(a,b) < \frac{1}{2}|a-b|$

  $\Leftrightarrow$

  $a \neq b$ & $a \neq 0$ & $b \neq 0$

D. & R. the arithmetic - geometric mean
= AGM

let

- $a, b \in$ real nr
- $a, b \geq 0$
- the sequences

$(a_0, a_1, a_2, \cdots)$

$(b_0, b_1, b_2, \cdots)$

are defined recursively as follows:

(rec def)

$a_0 = a$

$b_0 = b$

$a_{n+1} = GM(a_n, b_n)$

$b_{n+1} = AM(a_n, b_n)$

($n \in$ nonneg int var)

then

- $\min(a,b) \le a_1 \le a_2 \le a_3 \le \cdots \le b_3 \le b_2 \le b_1 \le \max(a,b)$

- $b_{n+1} - a_{n+1} \le \frac{1}{2}(b_n - a_n) \le \frac{1}{2^{n+1}}|a-b|$ $\quad$ (n ∈ pos int var)

- $\exists \lim_{n\to\infty} a_n$ & $\exists \lim_{n\to\infty} b_n$ & $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ $\quad$ (n ∈ pos int var)

- the arithmetic - geometric mean of a and b

$=_{dn}$ AGM(a, b)

$=_{df} \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ $\quad$ (n ∈ pos int var)

- $\min(a,b) \le GM(a,b) \le AGM(a,b) \le AM(a,b) \le \max(a,b)$

note AGM ← the capitalized initial letters of
arithmetic - geometric mean

T. (Gauss) an elliptic integral of the first kind
for the arithmetic - geometric mean

let

- $a, b \in$ pos real nr
- $a > b$
- $k^2 =_{df} \dfrac{a^2 - b^2}{a^2}$

then

- AGM (a, b)

$$= \frac{a\pi}{2} \div \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2 t^2)}}\, dt$$

$$= \frac{a\pi}{2} \div \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2 \sin^2\theta}}\, d\theta$$

(this integral is nonelementary)

D. the ubiquitous constant

$=_{ab}$ ubiq const

$=_{dn}$ U wh U $\leftarrow$ <u>u</u>biquitous

$$=_{df} \text{AGM}\left(1, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{2\sqrt{2}} \div \int_0^1 \frac{1}{\sqrt{(1-t^2)(2-t^2)}}\,dt$$

$$= \frac{\pi}{2\sqrt{2}} \div \int_0^{\pi/2} \frac{1}{\sqrt{2-\sin^2\theta}}\,d\theta$$

$$= \frac{\Gamma^2(3/4)}{\sqrt{\pi}}$$

$$= 0.84721\ 30848\cdots$$

☐ the three classical means
are

• the arithmetic mean = AM
• the geometric mean = GM
• the harmonic mean = HM

of a finite sequence of real numbers
that are defined on the following page;
these three means
are then subsumed under
the more general notion of mean
that is given next;
note that capitalized initials
are used for briefer denotation

• the arithmetic mean of real numbers

$a_1,\ a_2, \cdots,\ a_n$ wh $n \in$ pos int

$=_{dn}$ $AM(a_1,\ a_2, \cdots,\ a_n)$

$=_{df}$ $\frac{1}{n}(a_1 + a_2 + \cdots + a_n)$

• the geometric mean of nonnegative real numbers

$a_1,\ a_2, \cdots,\ a_n$ wh $n \in$ pos int

$=_{dn}$ $GM(a_1,\ a_2, \cdots,\ a_n)$

$= \sqrt[n]{a_1 a_2 \cdots a_n} \geq 0$

• the harmonic mean of nonzero real numbers

$a_1,\ a_2, \cdots,\ a_n$ wh $n \in$ pos int

$=_{dn}$ $HM(a_1,\ a_2, \cdots,\ a_n)$

$=_{df}$ the reciprocal of the arithmetic mean

of the reciprocals of $a_1,\ a_2, \cdots,\ a_n$

note $HM(a, b) = \dfrac{2ab}{a + b}$

D. & R. the general classical mean

let

- $n \in$ pos int
- $a = (a_1, a_2, \cdots, a_n) \in$ ordered n - tuple of nonnegative real numbers
- $r \in$ nonzero real number variable

then

- the classical mean of a with index r

$=_{dn} M_r(a) = M_r$ wh $M \leftarrow \underline{m}ean$

$$=_{df} \left(\frac{1}{n} \sum_{k=1}^{n} a_k^r \right)^{\frac{1}{r}}$$

(if $a_k = 0$ for some $k \in \underline{n}$ & $r < 0$,

then $M_r(a) =_{df} 0$)

&

it follows that

- $M_1(a) = AM(a)$
- $M_0(a) =_{df} \lim_{r \to 0} M_r(a) = GM(a)$
- $M_{-1}(a) = HM(a)$
- M_r is a weakly increasing function of

the real number variable r

- $\exists \lim_{r \to -\infty} M_r = \min(a)$
- $\exists \lim_{r \to +\infty} M_r = \max(a)$
- $\min(a) \le M_r \le \max(a) \quad (r \in \text{real nr})$
- $\min(a) \le HM(a) \le GM(a) \le AM(a) \le \max(a)$

(called
the arithmetic-geometric-harmonic mean inequality
= AGHMI)

&

furthermore

if the a_k ($k \in \underline{n}$) are all positive & not all equal, then

• M_r is a strictly increasing function of
the real number variable r

• $\min(a) < M_r < \max(a) \quad (r \in \text{real nr})$

• $\min(a) < HM(a) < GM(a) < AM(a) < \max(a)$

R. relating AM & GM
by the inverse functions:
the exponential function exp
&
the logarithm function log

• the exponential function
exp: $\mathbb{R} \to \mathbb{R}_+$
(mapping
the real line $\mathbb{R}$
one - to - one onto
the positive real ray $\mathbb{R}_+$)
carries
the arithmetic mean AM in $\mathbb{R}$
to
the geometric mean GM in $\mathbb{R}_+$
ie
$\exp \mathrm{AM}(a, b) = \mathrm{GM}(\exp a, \exp b) \quad (a, b \in \mathbb{R})$
$\exp \mathrm{AM}(a, b, c) = \mathrm{GM}(\exp a, \exp b, \exp c) \quad (a, b, c \in \mathbb{R})$
etc

• the logarithm function

log: $\mathbb{R}_+ \to \mathbb{R}$

(mapping

the positive real ray $\mathbb{R}_+$

one - to - one onto

the real line $\mathbb{R}$)

carries

the geometric mean GM in $\mathbb{R}_+$

to

the arithmetic mean AM in $\mathbb{R}$

ie

$$\log \mathrm{GM}(a, b) = \mathrm{AM}(\log a, \log b) \quad (a, b \in \mathbb{R}_+)$$

$$\log \mathrm{GM}(a, b, c) = \mathrm{AM}(\log a, \log b, \log c) \quad (a, b, c \in \mathbb{R}_+)$$

etc

R. $HM,\ GM,\ AM\ \in\ GP$

let

- $a, b\ \in$ real nr
- $a, b > 0$

then

- $GM(AM(a,b), HM(a,b))\ =\ GM(a,b)$
- $HM(a,b),\ GM(a,b),\ AM(a,b)$

are in geometric progression

with common ratio $= \dfrac{a+b}{2\sqrt{ab}}$

R. let

- $a, b \in$ real nr
- $a, b > 0$

then

- $0 \leq AM(a,b) - HM(a,b) \leq \frac{1}{2}|a-b|$
- $0 < AM(a,b) - HM(a,b) < \frac{1}{2}|a-b|$

  $\Leftrightarrow$

  $a \neq b$

R. a complicated way to obtain GM

let

- $a, b \in$ real nr
- $a, b > 0$
- the sequences

$(a_0,\ a_1,\ a_2,\ \cdots\ )$

$(b_0,\ b_1,\ b_2,\ \cdots\ )$

are defined recursively as follows:

(rec def)

$a_0 = a$

$b_0 = b$

$a_{n+1} = HM(a_n, b_n)$

$b_{n+1} = AM(a_n, b_n)$

($n \in$ nonneg int var)

then

- $ab = a_0 b_0 = a_1 b_1 = a_2 b_2 = \cdots$

- $\min(a,b) \le a_1 \le a_2 \le a_3 \le \cdots \le b_3 \le b_2 \le b_1 \le \max(a,b)$

- $b_{n+1} - a_{n+1} \le \frac{1}{2}(b_n - a_n) \le \frac{1}{2^{n+1}}|a-b|$ (n ∈ pos int var)

- $\exists \lim_{n\to\infty} a_n$ & $\exists \lim_{n\to\infty} b_n$

& $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = GM(a,b)$

(n ∈ pos int var)

R. conjectured origin of terms

• the term
geometric mean
likely came from the geometric context
in which a right triangle
with altitude to the hypotenuse
exhibits
three geometric means
involving
legs,
hypotenuse,
segments of the hypotenuse,
altitude

• the term
harmonic mean
likely came from the musical context
in which a vibrating stretched string
emitting a musical note
has its frequency of vibration
inversely proportional to its length;
note that
doubling the frequency (= dividing the length by 2)
raises the musical note an octave,
tripling the frequency (= dividing the length by 3)
raises the musical note by an octave and a fifth,
etc

Q & A. ¿ why does
mean (noun & adj)
mean
average ?

because
mean

↑

mene (Middle English) = middle

↑

moien (Old French)
= moyen (Modern French) = middle

↑

medianus (Late Latin)
= that which is in the middle

↑

medius (Latin) = middle

of course
mean
has other meanings too
as is customary with words