Math Medley

#45 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2001

GG45-1 (30)

500 Angell St #414
Providence RI 02906

GG45-2

□ unified product rule for derivatives

∆ to differentiate a product,
differentiate each factor separately
& add

∆ applications

- product of any number of real/complex functions
- scalar product of two vectors
- vector product of two 3-vectors
- triple scalar product of three 3-vectors
- triple vector product of three 3-vectors
- multiple vector products of 3-vectors
- determinant of any order (op on rows or cols)
- etc

∆ the uniformity of this product rule
comes from the multilinearity
of these products

T. let

$f \in$ real – valued function on a real interval

then

$f \in$ differentiable

$\Downarrow$

$f \in$ continuous

$\Downarrow$

$f \in$ integrable

&

all converses fail

D. a Leibniz series

$=_{df}$ an alternating series

$a_0 - a_1 + a_2 - a_3 + \cdots$

of positive real numbers

$a_0,\ a_1,\ a_2,\ a_3,\ \cdots$

st

$a_n \geq a_{n+1}$ ($n \in$ nonneg int var)

&

$a_n \to 0$ as $n \to \infty$

T. every Leibniz series converges

T. the harmonic series diverges

P. otherwise

s

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

$$= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots$$

$$> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

$= s$

& $s > s$

$\therefore$ the harmonic series diverges

QED

□ an interesting inequality

T. $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n} \quad (n \in \text{int} \geq 2)$

P. note that

$\dfrac{1}{\sqrt{k}} > \sqrt{k} - \sqrt{k-1}$ for $k \geq 2$

&

a telescoping sum results from substitution

QED

□ a theorem
whose proof from the analytic POV
is not immediate
but which is evident from the geometric POV

T. let
- $a, b \in$ real nos st $a < b$
- $y = y(x),\ 0 \leq x \leq a$

&

$x = x(y),\ 0 \leq y \leq b$

$\in$ inverse strictly increasing continuous real functions

then
- $\int_0^a y\,dx + \int_0^b x\,dy = ab$

T. the fundamental theorem for space curves

let

- s = arclength
- κ = curvature
- τ = torsion
- $\kappa = \kappa(s) \geq 0$ & $\tau = \tau(s)$

$\in$ given continuous real - valued functions on a real interval

then

- there exists a class C^2 space curve,

unique up to rigid motion,

with the given curvature $\kappa = \kappa(s)$

&

with the given torsion $\tau = \tau(s)$

☐ four forms of the Dirichlet
box / drawer / pigeonhole principle

(1) if $n+1$ objects are placed in n boxes
where n is a positive integer,
then at least one box contains at least two objects

(2) if X and Y are sets of any cardinality
such that $\operatorname{crd} X > \operatorname{crd} Y$, and if $f: X \to Y$,
then there exists $y \in Y$ such that $\operatorname{crd} f^{-1}(y) > 1$

(3) if infinitely many objects
are placed in finitely many boxes,
then at least one box contains infinitely many objects

(4) if X is an infinite set, if Y is a finite set, and if $f: X \to Y$,
then there exists $y \in Y$ such that $f^{-1}(y)$ is an infinite set

(2) is a set - theoretic generalization of (1);
(4) is a set - theoretic equivalent of (3)

Question.
Is the following statement mathematically true?
If there are more trees in the world
than there are leaves on any one tree,
then there are at least two trees
with the same number of leaves.

Answer.
Not as it stands.
It is true under three additional conditions, namely:
there are only finitely many trees in the world,
there is at least one tree in the world,
every tree has at least one leaf.

□ Lagrange's identity for vectors

T. Lagrange's identity for two 2 - vectors

let

$\mathbf{a}, \mathbf{b} \in$ 2 - vectors over a com ring

then

$$|\ \mathbf{a}, \mathbf{b}|^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{vmatrix}$$

T. Lagrange' s identity for two 3 - vectors

let

$\mathbf{a}, \mathbf{b} \in 3$ - vectors over a com ring

then

$$(\mathbf{a} \times \mathbf{b})^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{vmatrix}$$

T. Lagrange's identity for two vectors

let

$\mathbf{a}, \mathbf{b} \in$ n - vectors over a com ring ($n \in$ pos int)

then

• in scalar notation

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}^2 = \begin{vmatrix} \sum_{i=1}^{n} a_i^2 & \sum_{i=1}^{n} a_i b_i \\ \sum_{i=1}^{n} a_i b_i & \sum_{i=1}^{n} b_i^2 \end{vmatrix}$$

• in vector notation

$$\frac{1}{2}(\mathbf{ab} - \mathbf{ba})^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix}$$

T. Lagrange' s identity for four 2 - vectors

let

$\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in$ 2 - vectors over a com ring

then

$$|\mathbf{a}, \mathbf{b}| \cdot |\mathbf{c}, \mathbf{d}| = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

T. Lagrange' s identity for four 3 - vectors

let

$\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in$ 3 - vectors over a com ring

then

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

T. Lagrange's identity for four vectors

let

$\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in$ n - vectors over a com ring (n $\in$ pos int)

then

• in scalar notation

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \cdot \begin{vmatrix} c_i & c_j \\ d_i & d_j \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{n} a_i c_i & \sum_{i=1}^{n} a_i d_i \\ \sum_{i=1}^{n} b_i c_i & \sum_{i=1}^{n} b_i d_i \end{vmatrix}$$

• in vector notation

$$(\mathbf{ab} - \mathbf{ba}) \cdot (\mathbf{cd} - \mathbf{dc}) = 2 \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

T. Cauchy - Riemann equations

= CRE

let

- $x, y, u, v \in$ real var
- $z, w \in$ complex var
- $x + iy = z$
- $u + iv = w$
- $w = w(z)$
- $\exists \dfrac{dw}{dz}$

then

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

P. in three parts

(1) $\dfrac{dw}{dz}$

$= \dfrac{\partial w}{\partial x}$

$= \dfrac{\partial}{\partial x}(u + iv)$

$= u_x + iv_x$

(2) $\dfrac{dw}{dz}$

$= \dfrac{\partial w}{\partial (iy)}$

$= \dfrac{\partial}{\partial (iy)}(u + iv)$

$= \dfrac{1}{i}\dfrac{\partial}{\partial y}(u + iv)$

$= \dfrac{1}{i}\left(u_y + iv_y\right)$

$= v_y - iu_y$

(3) $u_x + i v_x$

$$= \frac{dw}{dz}$$

$$= v_y - i u_y$$

$\therefore\ u_x = v_y \ \&\ u_y = -v_x$

qed

K. the Jacobian of (u, v) wrt (x, y)

$$= \det \frac{\partial(u,v)}{\partial(x,y)}$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= u_x v_y - u_y v_x$$

$$= {u_x}^2 + {v_x}^2$$

$$= {u_y}^2 + {v_y}^2$$

$$= {u_x}^2 + {u_y}^2$$

$$= {v_x}^2 + {v_y}^2$$

$$= \left|\frac{dw}{dz}\right|^2$$

$$\geq 0$$

□ the law of inclusion - exclusion for finite sets

Δ law of inclusion - exclusion
for two finite sets A, B

- $\operatorname{crd}(A \cup B) = \operatorname{crd} A + \operatorname{crd} B - \operatorname{crd}(A \cap B)$

Δ law of inclusion - exclusion
for three finite sets A, B, C

- $\operatorname{crd}(A \cup B \cup C)$

$= \operatorname{crd} A + \operatorname{crd} B + \operatorname{crd} C$

$- \operatorname{crd}(A \cap B) - \operatorname{crd}(A \cap C) - \operatorname{crd}(B \cap C)$

$+ \operatorname{crd}(A \cap B \cap C)$

Δ law of inclusion - exclusion

for n finite sets $A_1,\ A_2,\ \cdots\ ,\ A_n \quad (n \in \text{pos int})$

$$\bullet\ \operatorname{crd} \bigcup_{i=1}^{n} A_i$$

$$= \sum_{i=1}^{n} \operatorname{crd} A_i$$

$$- \sum_{\substack{i,j=1 \\ i<j}}^{n} \operatorname{crd}(A_i\, A_j)$$

$$+ \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} \operatorname{crd}(A_i\, A_j\, A_k)$$

$$- \cdots$$

$$+ (-1)^{n+1} \operatorname{crd}(A_1\, A_2 \cdots A_n)$$

□ the general law of inclusion-exclusion
not only holds for finite cardinality
but also holds for the measure
of finitely many sets of finite measure
in a finitely additive measure space
and also holds for the indicator
applied to finitely many sets (these sets finite or not);
the cases of 2 & 3 sets
are given below

□ the law of inclusion - exclusion
for the measure
of two sets A, B
&
of three sets A, B, C,
all of finite measure in a finitely additive measure space

- $m(A \cup B) = mA + mB - m(A \cap B)$

- $m(A \cup B \cup C)$

$= mA + mB + mC$

$- m(A \cap B) - m(A \cap C) - m(B \cap C)$

$+ m(A \cap B \cap C)$

□ the law of inclusion - exclusion
for the indicator
applied to two sets A, B
&
applied to three sets A, B, C

- $I(x, A \cup B) = I(x, A) + I(x, B) - I(x, A \cap B)$

- $I(x, A \cup B \cup C)$
$= I(x, A) + I(x, B) + I(x, C)$
$- I(x, A \cap B) - I(x, A \cap C) - I(x, B \cap C)$
$+ I(x, A \cap B \cap C)$

□ relative strengths
of the inclusion-exclusion laws

the law for the indicator

implies

the law for measure

implies

the law for cardinality

□ the Heisenberg indeterminacy / uncertainty principle applies to a particle

$\Delta q \ \Delta p \geq h / 2\pi$
&
$\Delta t \ \Delta E \geq h / 2\pi$
wh
Δ = error in measurement of
q = position coordinate
p = momentum
t = time
E = energy
h = Planck's constant

• bioline
Werner Karl Heisenberg
1901 - 1976
German
theoretical physicist

□ a guiding principle

• the drive toward brevity:

to say more & more

in less & less

time & space

because

we have lots to say

&

we have little

time & space

to say it in

□ the exposition of mathematics
= mathematical exposition
= mathematics exposition

is

an art form

&

should be

a work of art