That Sterling Formula:
Stirling's Formula

#40 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2001

500 Angell St #414
Providence RI 02906

GG40-2

□ the factorial n! of a positive integer n
grows rapidly with n;
eg in base ten expansion
10! has 7 digits
&
100! has 158 digits;
how to readily approximate factorials
is a question answered by Stirling's formula;
Stirling's formula replaces (in approximation)
multiplication by changing factors
with repeated multiplication by the same factor
ie exponential functions
(which themselves are in principle
easier to approximate)

□ how to understand
that sterling formula
Stirling's formula

D. asymptotic functions
let
- f, g: $\mathbb{P} \to \mathbb{R}$
- $n \in$ var $\mathbb{P}$

then
- f(n) is asymptotic to g(n) as n goes to infinity

$=_{dn}$ $f(n) \sim g(n)$ as $n \to \infty$
$=_{df}$ $g(n) \neq 0$ for n suff large
&
$$\exists \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

Δ here are four forms of Stirling' s formula

- $n! \sim \sqrt{2\pi n}\ e^{-n}\, n^{n}$ as $n \to \infty$

- $n! \sim \sqrt{2\pi}\ e^{-n}\, n^{n+1/2}$ as $n \to \infty$

- $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}$ as $n \to \infty$

- $n! \sim \sqrt{2\pi e}\left(\frac{n}{e}\right)^{n+\frac{1}{2}}$ as $n \to \infty$

wh $n \in$ pos int var

- all four RHS are equal

- note the values

$\sqrt{2\pi} = 2.5066+$

$\sqrt{2\pi e} = 4.1327+$

C. consider the form of Stirling' s formula

$$n! \sim \sqrt{2\pi e}\left(\frac{n}{e}\right)^{n+\frac{1}{2}} \quad \text{as} \;\; n \to \infty \quad (n \in \text{pos int var})$$

this formula has the form

$$n! \sim k\, f(n)^{g(n)} \quad \text{as} \;\; n \to \infty$$

wh

$$k = \sqrt{2\pi e} \;\approx\; 4.1327+$$

is a very pretty constant

and

$$f(n) = \frac{n}{e}$$

&

$$g(n) = n + \frac{1}{2}$$

are very simple linear functions of n

note that

$$f(n) = \frac{n}{e}$$

is a kind of a measure

of the average size of the n factors in n!

& that

$$g(n) = n + \frac{1}{2}$$

is almost the number n of the factors in n!

thus

replacing a factorial n!

by a power $f(n)^{g(n)}$ (times a constant)

C. another way of looking at Stirling' s formula
is this:

(1) n! is somewhat like n^n
but that' s too large

(2) so correct downward
by multiplying by e^{-n}
so that
n! is like $e^{-n} n^n$
but that' s too small

(3) so correct upward
by multiplying by $\sqrt{2 \pi n}$
so that
n! is like $\sqrt{2 \pi n}\, e^{-n} n^n$
which hits the spot
ie
the ratio goes to 1
as n goes to ∞

Δ to attain an idea of the accuracy
of Stirling' s approximation

$$S_n = \sqrt{2\pi n}\, e^{-n} n^n \quad (n \in \text{pos int})$$

examine the inequalities

- $S_n < n! < S_n \exp\frac{1}{12n}$

- $1 < \frac{n!}{S_n} < \exp\frac{1}{12n}$

wh $n \in$ pos int;
for each positive integer n
there exists a real number $\vartheta(n)$ st
$0 < \vartheta(n) < 1$
&

$$n! = S_n \exp\frac{\vartheta(n)}{12n}$$

C. Stirling's formula
extends to
Stirling's series
which approximates
the gamma function $\Gamma(z)$
which is a generalization to
an independent complex variable z
of the factorial function n!
of a nonegative integer variable n;
specifically
$\Gamma(n+1) = n!$ for $n \in$ nonneg int

HN. Stirling's formula is actually due to De Moivre

• James Stirling
1692-1770
Scottish
analyst, industrial manager

• Abraham De Moivre
1667-1754
French-English
analyst, probabilist, statistician