A Parade of Integrals

#39 of Gottschalk’s Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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GG39-2

□ the notation of integration:
what various kinds of integrals look like;
a sampling

Δ indefinite integral

- $\int f(x)\,dx$

Δ definite integral

- $\int_a^b f(x)\,dx$
- $\int_I f$

Δ improper integrals with infinite range

- $\int_a^\infty f(x)\,dx$
- $\int_{-\infty}^b f(x)\,dx$
- $\int_{-\infty}^\infty f(x)\,dx$

Δ iterated integrals

- $\int_a^b \left[\int_{x=\alpha(y)}^{x=\beta(y)} f(x,y)\,dx \right] dy$

- $\int_a^b \left[\int_{y=\gamma(z)}^{y=\delta(z)} \left[\int_{x=\alpha(y,z)}^{x=\beta(y,z)} f(x,y,z)\,dx \right] dy \right] dz$

Δ multiple integrals

- $\iint_D f(x,y)\,dx\,dy$

- $\iiint_D f(x,y,z)\,dx\,dy\,dz$

Δ line integrals

- $\int_C f(x)\,dx$
- $\int_C f(x,y)\,dx + g(x,y)\,dy$
- $\int_C f(x,y,z)\,dx + g(x,y\,z)\,dy + h(x,y,z)\,dz$
- $\int_C f(x)\,ds$
- $\int_C f(x,y)\,ds$
- $\int_C f(x,y,z)\,ds$
- $\int_C f(\mathbf{r})\,ds$

Δ line integrals around a closed curve

- $\oint_C f(x)\,dx$

- $\oint_C f(x,y)\,dx + g(x,y)\,dy$

- $\oint_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$

- $\oint_C f(x)\,ds$

- $\oint_C f(x,y)\,ds$

- $\oint_C f(x,y,z)\,ds$

- $\oint_C f(\mathbf{r})\,ds$

Δ surface integrals

- $$\iint_S f(x,y,z)\,dy\,dz + g(x,y,z)\,dz\,dx + h(x,y,z)\,dx\,dy$$

$$= \int_S f(x,y,z)\,dy\,dz + g(x,y,z)\,dz\,dx + h(x,y,z)\,dx\,dy$$

- $$\iint_S f(x,y,z)\,dS = \int_S f(x,y,z)\,dS$$

Δ surface integrals over a closed surface

- $$\oiint_S f(x,y,z)\,dy\,dz + g(x,y,z)\,dz\,dx + h(x,y,z)\,dx\,dy$$

$$= \oint_S f(x,y,z)\,dy\,dz + g(x,y,z)\,dz\,dx + h(x,y,z)\,dx\,dy$$

- $$\oiint_S f(x,y,z)\,dS = \oint_S f(x,y,z)\,dS$$

Δ area integrals

- $$\iint\limits_A f(x,y)\,dx\,dy = \int_A f(x,y)\,dx\,dy$$

$$= \iint\limits_A f(x,y)\,dA = \int_A f(x,y)\,dA$$

Δ volume integrals

- $$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \int_V f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint\limits_V f(x,y,z)\,dV = \int_V f(x,y,z)\,dV$$

Δ vector integrals

- $\int_D \mathbf{f}(x)\,dx$

- $\int_D f(\mathbf{r})\,d\mathbf{r}$

- $\int_D \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r}$

- $\int_D \mathbf{f}(\mathbf{r}) \times d\mathbf{r}$

Δ Stieltjes integral

- $\int_a^b f(x)\,d\alpha(x)$

Δ integrals of exterior differential forms / polynomials

- $\int_D f(x)\,dx$

- $\int_D f(x, y)\,dx \wedge dy$

- $\int_D f(x, y, z)\,dx \wedge dy \wedge dz$

Δ complex indefinite integral

- $\int f(z)\,dz$

Δ complex line integral

- $\int_C f(z)\,dz$

Δ integral over a measure space

- $\int_S f(x)\,dm(x)$

Δ Stokes' formula / law / theorem

• $\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$

for an exterior differential form ω

on a differentiable manifold Ω

Δ integrals according to kind of integration

• Riemann integral
• Lebesgue integral
• Riemann - Stieltjes integral
• Lebesgue - Stieltjes integral
• Cauchy principal - value integral
• upper / lower Darboux integral
• Banach integral
• gauge integral

etc

Δ syntactic origins of integral & differential notations

• the integral sign

$\int$

is a stylized elongated letter ess Ss

from the initial letter

of the Latin noun ' summa' meaning ' sum'

• the differential sign

d

is the lowercase letter dee

from the initial letter

of the Latin noun ' differentia' meaning ' difference'

• both notations are due to Leibniz

□ some just - for - fun definite integrals
involving the exponential function

- $$\int_0^\infty \frac{1}{e^x + 1}\,dx = \log 2$$

- $$\int_0^\infty \frac{x}{e^x + 1}\,dx = \frac{\pi^2}{12}$$

- $$\int_0^\infty \frac{x}{e^x - 1}\,dx = \frac{\pi^2}{6}$$

- $$\int_0^\infty e^{-x}\,dx = 1$$ (exponential integral)

- $$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$$ (probability integral)

□ some just - for - fun definite integrals
involving the logarithm function

- $$\int_0^1 \frac{\log x}{x+1}\,dx = -\frac{\pi^2}{12}$$

- $$\int_0^1 \frac{\log(x+1)}{x}\,dx = \frac{\pi^2}{12}$$

- $$\int_0^\infty e^{-x} \log x \, dx = \int_0^1 \log\log\frac{1}{x}\,dx = -\gamma$$

wh γ = Euler's constant = 0.57721 56649 +

(dampened log integral on left,
log log integral on right)

- $$\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \int_0^{\frac{\pi}{2}} \log \cos x \, dx = -\frac{\pi}{2} \log 2$$

- $$\int_0^{\frac{\pi}{2}} \log \tan x \, dx = \int_0^{\frac{\pi}{2}} \log \cot x \, dx = 0$$

- $$\int_0^{\frac{\pi}{2}} \log \sec x \, dx = \int_0^{\frac{\pi}{2}} \log \csc x \, dx = \frac{\pi}{2} \log 2$$

(called log sine, log cosine, etc, integrals)

□ some integrals involving sine & cosine

- $\int_{-\infty}^{\infty} \sin x^2 \, dx = \int_{-\infty}^{\infty} \cos x^2 \, dx = \sqrt{\frac{\pi}{2}}$

- $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$

- $\int_{0}^{\infty} e^{-x} \sin x \, dx = \frac{1}{2}$

(dampened sine integral)

- $\int_{0}^{\infty} e^{-x} \cos x \, dx = \infty$

(dampened cosine integral?)

□ some functions defined by integrals

• Gamma function integral

$$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx$$

wh $n \in$ real nr > -1

• Beta function integral

$$B(m+1,\ n+1) = \int_0^1 x^m (1-x)^n dx$$

wh $m, n \in$ real nr > -1

• Fresnel sine integral

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} dt = \sqrt{\frac{2}{\pi}} \int_0^x \sin t^2 dt$$

• Fresnel cosine integral

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt = \sqrt{\frac{2}{\pi}} \int_0^x \cos t^2 dt$$

• exponential integral function

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt$$

• logarithmic integral function

$$\mathrm{Li}(x) = \int_{0}^{x} \frac{dt}{\log t}$$

• sine integral function

$$\mathrm{Si}(x) = \int_{0}^{x} \frac{\sin t}{t}\, dt$$

• cosine integral function

$$\mathrm{Ci}(x) = -\int_{x}^{\infty} \frac{\cos t}{t}\, dt$$