The Ackermann Number Explosion

#36 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2001

GG36-1 (18)

500 Angell St #414
Providence RI 02906

GG36-2

☐ the Ackermann number explosion

• Ackermann's function
of two independent nonnegative integer variables
and which is positive-integer-valued
is here presented equivalently
(and I think more clearly)
as a sequence of positive-integer-valued functions
of a single nonnegative integer variable

$$f_0(x),\ f_1(x),\ f_2(x),\ \cdots \quad (x \in \text{nonneg int var})$$

Ackermann's sequence of functions
is defined by a double recursion
(one recursion equation on n, from n to n+1
& one recursion equation on x, from x to x+1)
as follows:

$$f_0(x) = x + 1$$

$$f_{n+1}(0) = f_n(1)$$

$$f_{n+1}(x+1) = f_n(f_{n+1}(x))$$

(rec def; n, x $\in$ nonneg int var)

• the first two equations
in the recursive definition
are just to get things started;
it's the third equation
that provides the bombshell growth;
it makes one step of one sequence
provide the growth of an entire initial segment
of the preceding sequence

• the growth and the size of $f_n(x)$
as n and x get larger
are phenomenal;
to illustrate this growth and size,
the number of digits
in the base 10 expansion of $f_4(3)$
is vastly more than
the estimated number of particles (however defined)
in the observable universe;
Ackermann's diagonal function $f_n(n)$
of a single nonnegative integer variable n
and which is positive-integer-valued
is an example of a computable function
that is not primitive recursive

• it follows from the definition that

$$f_0(x) = x + 1$$

$$f_1(x) = x + 2$$

$$f_2(x) = 2x + 3$$

$$f_3(x) = 2^{x+3} - 3$$

$$f_4(x) = (\text{exponential tower of all 2's with } x + 3 \text{ stories}) - 3$$

to express $f_5(x)$ and higher functions in closed form customary mathematical notation is inadequate; something beyond needs to be (and has been) designed; see below

• the Knuth up-arrow notation
permits the notational continuation of the sequence
addition, multiplication, exponentiation, etc;
it is described as follows
where m and n are positive integers;
association/parenthesizing on the right is understood;
the up-arrow ↑ may be read simply ‘up’

$$mn = m + m + \cdots + m \text{ (n terms) } \text{ (0 arrows)}$$

$$m \uparrow n = m \times m \times \cdots \times m \text{ (n terms) (1 arrow)}$$

$$m \uparrow\uparrow n = m \uparrow m \uparrow \cdots \uparrow m \text{ (n terms) (2 arrows)}$$

$$m \uparrow\uparrow\uparrow n = m \uparrow\uparrow m \uparrow\uparrow \cdots \uparrow\uparrow m \text{ (n terms) (3 arrows)}$$

etc

note

$$mn = \text{ordinary multiplication}$$
$$= \text{repeated addition}$$

$$m \uparrow n = m^n = \text{ordinary exponentiation}$$
$$= \text{repeated multiplication}$$

$$m \uparrow\uparrow n = \text{power tower of n stories of m}$$
$$= \text{repeated exponentiation}$$

• it may be helpful to describe this notation
in an informal way;
consider two positive integers m and n
separated by a finite sequence of up-arrows;
this notation stands for a positve integer
that is calculated as follows;
the process is recursive,
reducing the number of arrows in a gap
by one at each step
until (in principle) no arrow remains
and only ordinary multiplication remains;
to eliminate the terminal arrow say
between m and n,
write down n terms of m with n - 1 gaps;
fill each of the n - 1 gaps with one less arrow than before;
associate on the right;
repeat (in principle perhaps)
until the last arrow is eliminated
and only juxtaposition remains
which is then ordinary multiplication,
or equivalently until only one arrow remains
which then specifies ordinary exponentiation

• power towers ito up - arrows;

up - arrows flatten power towers

a, b, c, $\cdots \in$ pos int (say)

$\Rightarrow$

$a = a$ (power tower of 1 story)

$a^b = a \uparrow b$ (power tower of 2 stories)

$a^{b^c} = a \uparrow b \uparrow c$ (power tower of 3 stories)

etc

in general

an n - story power tower

needs n – 1 up – arrows & n terms

to be flattened

wh $n \in$ pos int;

if all n stories are the same a,

then $a \uparrow\uparrow n$ will do

• we give some illustrative examples
chosen from m, n = 2, 3, 4

$$2 \uparrow 2 = 2(\times)2 = 2^2 = 4$$

$$2 \uparrow 3 = 2(\times)2(\times)2 = 2^3 = 8$$

$$2 \uparrow 4 = 2(\times)2(\times)2(\times)2 = 2^4 = 16$$

$$2 \uparrow\uparrow 2 = 2 \uparrow 2 = 4$$

$$2 \uparrow\uparrow 3 = 2 \uparrow 2 \uparrow 2 = 2 \uparrow (2 \uparrow 2) = 2 \uparrow 4 = 16$$

$$2 \uparrow\uparrow 4 = 2 \uparrow 2 \uparrow 2 \uparrow 2 = 2 \uparrow 2 \uparrow (2 \uparrow 2)$$
$$= 2 \uparrow 2 \uparrow 4 = 2 \uparrow (2 \uparrow 4)$$
$$= 2 \uparrow 16 = 2^{16} = 65536$$

$$2 \uparrow\uparrow\uparrow 2 = 2 \uparrow\uparrow 2 = 2 \uparrow 2 = 4$$

$$2 \uparrow\uparrow\uparrow 3 = 2 \uparrow\uparrow 2 \uparrow\uparrow 2 = 2 \uparrow\uparrow (2 \uparrow\uparrow 2) = 2 \uparrow\uparrow 4$$

$$= 2 \uparrow 2 \uparrow 2 \uparrow 2 = 2 \uparrow 2 \uparrow (2 \uparrow 2)$$

$$= 2 \uparrow 2 \uparrow 4 = 2 \uparrow (2 \uparrow 4)$$

$$= 2 \uparrow 16 = 2^{16} = 65536$$

$$2 \uparrow\uparrow\uparrow 4 = 2 \uparrow\uparrow 2 \uparrow\uparrow 2 \uparrow\uparrow 2 = 2 \uparrow\uparrow 2 \uparrow\uparrow (2 \uparrow\uparrow 2)$$

$$= 2 \uparrow\uparrow 2 \uparrow\uparrow 4 = 2 \uparrow\uparrow (2 \uparrow\uparrow 4) = 2 \uparrow\uparrow 65536$$

$$= 2 \uparrow 2 \uparrow \cdots \uparrow 2 \quad \text{(65536 terms)}$$

$$3 \uparrow 2 = 3(\times)3 = 3^2 = 9$$

$$3 \uparrow 3 = 3(\times)3(\times)3 = 3^3 = 27$$

$$3 \uparrow 4 = 3(\times)3(\times)3(\times)3 = 3^4 = 81$$

$$3 \uparrow\uparrow 2 \;=\; 3 \uparrow 3 \;=\; 27$$

$$3 \uparrow\uparrow 3 \;=\; 3 \uparrow 3 \uparrow 3 \;=\; 3 \uparrow (3 \uparrow 3) \;=\; 3 \uparrow 27 \;=\; 3^{27}$$

$$3 \uparrow\uparrow 4 \;=\; 3 \uparrow 3 \uparrow 3 \uparrow 3 \;=\; 3 \uparrow 3 \uparrow (3 \uparrow 3)$$

$$= \; 3 \uparrow 3 \uparrow 27 \;=\; 3 \uparrow (3 \uparrow 27)$$

$$= \; 3 \uparrow 3^{27} \;=\; 3^{3^{27}}$$

$$3 \uparrow\uparrow\uparrow 2 \;=\; 3 \uparrow\uparrow 3 \;=\; 3^{27}$$

$$3 \uparrow\uparrow\uparrow 3 \;=\; 3 \uparrow\uparrow 3 \uparrow\uparrow 3 \;=\; 3 \uparrow\uparrow (3 \uparrow\uparrow 3) \;=\; 3 \uparrow\uparrow 3^{27}$$

$$= \; 3 \uparrow 3 \uparrow \cdots \uparrow 3 \quad (3^{27} \text{ terms})$$

$$3 \uparrow\uparrow\uparrow 4 \;=\; 3 \uparrow\uparrow 3 \uparrow\uparrow 3 \uparrow\uparrow 3 \;=\; 3 \uparrow\uparrow 3 \uparrow\uparrow (3 \uparrow\uparrow 3)$$

$$= \; 3 \uparrow\uparrow 3 \uparrow\uparrow 3^{27} \;=\; 3 \uparrow\uparrow (3 \uparrow\uparrow 3^{27})$$

$$= \; 3 \uparrow 3 \uparrow \cdots \uparrow 3 \quad (3 \uparrow\uparrow 3^{27} \text{ terms})$$

• ito the up-arrow notation
the Ackermann sequence of functions
is

$$f_1(x) = 2 + (x+3) - 3$$

$$f_2(x) = 2(x+3) - 3$$

$$f_3(x) = 2 \uparrow (x+3) - 3$$

$$f_4(x) = 2 \uparrow\uparrow (x+3) - 3$$

$$f_5(x) = 2 \uparrow\uparrow\uparrow (x+3) - 3$$

etc

note that the number of arrows
is two less than the index

• in thinking about
the Ackermann sequence of functions
it may be helpful at times to consider
the sequence of functions as
an infinite matrix of positive integers
(except for the entry of 0 in the corner)
as follows:

the 1st row = the values of x from x = 0 onward
the 2nd row = the values of $f_0(x)$ from x = 0 onward
the 3rd row = the values of $f_1(x)$ from x = 0 onward
etc

thus

x:	0	1	2	3	4	5	6	7	8	9	etc= x
f_0:	1	2	3	4	5	6	7	8	9	10	etc= x+1
f_1:	2	3	4	5	6	7	8	9	10	11	etc= x+2
f_2:	3	5	7	9	11	13	15	17	19	21	etc=2x+3
f_3:	5	13	29	61	125						etc=$2^{x+3}-3$

etc

the first entry for each f
is the second entry in the line above;
to get a later entry for a given f,
look at the value of the entry before
and take the entry from the line above
at that same value for x

• bioline

Wilhelm Ackermann

1896-1962

German

mathematical logician;

student and collaborator of Hilbert;

first defined an earlier version

of the present Ackermann function in 1928