Quadratic Polynomials & Equations

#35 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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□ factoring & solving
quadratic polynomials & equations

Δ in the following discussion
criteria are given ito the coefficients a, b, c for
(1) the factorability
of the quadratic polynomial
$a x^2 + b x + c \quad (a \neq 0)$
(2) the solvability
of the quadratic equation
$a x^2 + b x + c = 0 \quad (a \neq 0)$
over
• the integer ring $\mathbb{Z}$
• the rational field $\mathbb{Q}$
• the real field $\mathbb{R}$
• the complex field $\mathbb{C}$

Δ also there is given the explicit factorization / solution
of the quadratic polynomial / equation ito the coefficients
when it is factorable / solvable over a particular system

Δ theorems on factoring
quadratic = second degree
polynomials in one variable
into two
linear = first degree
polynomial factors

Q. ¿ when is a quadratic polynomial factorable ?
it all depends on what you want the coefficients to be

D. discriminant

let

- $R \in$ ring
- $a, b, c \in R$ wh $a \neq 0$
- $x \in \text{var}\, R$

then

- the discriminant

of the polynomial $a\,x^2 + b\,x + c$

and

of the equation $a\,x^2 + b\,x + c = 0$

over R

$=_{dn}$ $D(a, b, c) = D$ wh $D \leftarrow \underline{d}$iscriminant

$=_{df}$ $b^2 - 4\,a\,c$

T. factorability over $\mathbb{Z}$

let

- a, b, c $\in \mathbb{Z}$ wh $a \neq 0$
- x $\in$ var $\mathbb{Z}$

then

- $a x^2 + b x + c \in$ fct / $\mathbb{Z}$

$\Leftrightarrow$

D $\in$ sqr in $\mathbb{Z}$

T. factorability over $\mathbb{Q}$

let

- a, b, c $\in \mathbb{Q}$ wh $a \neq 0$
- x $\in$ var $\mathbb{Q}$

then

- $a x^2 + b x + c \in$ fct / $\mathbb{Q}$

$\Leftrightarrow$

D $\in$ sqr in $\mathbb{Q}$

T. factorability over $\mathbb{R}$

let

- $a, b, c \in \mathbb{R}$ wh $a \neq 0$
- $x \in$ var $\mathbb{R}$

then

- $a x^2 + b x + c \in$ fct / $\mathbb{R}$

$\Leftrightarrow$

$D \geq 0$

T. factorability over $\mathbb{C}$

let

- $a, b, c \in \mathbb{C}$ wh $a \neq 0$
- $x \in$ var $\mathbb{C}$

then

- $a x^2 + b x + c \in$ always fct / $\mathbb{C}$

Q. ¿ when is a quadratic polynomial a perfect square ?
it all depends on what you want the coefficients to be

D. polynomial ring

let
- R ∈ ring
- x ∈ indeterminate

then
- the polynomial ring in x over R

$=_{dn}$ R[x]

$=_{rd}$ R bracket x

$=_{df}$ the ring of all polynomials in x
with coefficients from R

T. perfect squares in $\mathbb{Z}[x]$

let

- $a, b, c \in \mathbb{Z}$ wh $a \neq 0$
- $x \in$ var $\mathbb{Z}$

then

- $a x^2 + b x + c \in$ sqr in $\mathbb{Z}[x]$

$\Leftrightarrow$

$D = 0$ & $a, c \in$ sqr in $\mathbb{Z}$

T. perfect squares in $\mathbb{Q}[x]$

let

- $a, b, c \in \mathbb{Q}$ wh $a \neq 0$
- $x \in \text{var}\,\mathbb{Q}$

then

- $a x^2 + b x + c \in$ sqr in $\mathbb{Q}[x]$

$\Leftrightarrow$

$D = 0$ & $a, c \in$ sqr in $\mathbb{Q}$

T. perfect squares in $\mathbb{R}[x]$

let

- $a, b, c \in \mathbb{R}$ wh $a \neq 0$
- $x \in \text{var}\, \mathbb{R}$

then

- $a x^2 + b x + c \in$ sqr in $\mathbb{R}[x]$

$\Leftrightarrow$

$D = 0$ & $a > 0$ & $c \geq 0$

T. perfect squares in $\mathbb{C}[x]$

let

- a, b, c $\in \mathbb{C}$ wh $a \neq 0$
- x $\in$ var $\mathbb{C}$

then

- $a x^2 + b x + c \in$ sqr in $\mathbb{C}[x]$

$\Leftrightarrow$

$D = 0$

Δ the quadratic formula
over $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ in turn
which gives
in a specified number system
all solutions
of the quadratic equation
explicitly ito the coefficients
if any solution exists

Q. ¿ when is a quadratic equation solvable ?
it all depends on what number system
you want to solve it in

T. the quadratic formula over $\mathbb{C}$

let

- $a,\ b,\ c \in \mathbb{C}$ wh $a \neq 0$
- $x \in$ var $\mathbb{C}$

then

- $a x^2 + b x + c = 0 \quad (\exists\ x \in \mathbb{C})$

- $a x^2 + b x + c = 0$

$\Leftrightarrow$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (\forall\ x \in \mathbb{C})$$

T. the quadratic formula over $\mathbb{R}$

let

- $a, b, c \in \mathbb{R}$ wh $a \neq 0$
- $x \in$ var $\mathbb{R}$

then

- $a x^2 + b x + c = 0 \quad (\exists x \in \mathbb{R})$

$\Leftrightarrow$

$b^2 - 4 a c \geq 0$

- $a x^2 + b x + c = 0$

$\Leftrightarrow$

$b^2 - 4 a c \geq 0$

&

$$x = \frac{-b \pm \sqrt{b^2 - 4 a c}}{2a} \quad (\forall x \in \mathbb{R})$$

T. the quadratic formula over $\mathbb{Q}$

let

- $a, b, c \in \mathbb{Q}$
- $x \in \text{var}\,\mathbb{Q}$

then

- $a x^2 + b x + c = 0 \quad (\exists x \in \mathbb{Q})$

$\Leftrightarrow$

$b^2 - 4ac \in \text{sqr in } \mathbb{Q}$

- $a x^2 + b x + c = 0$

$\Leftrightarrow$

$b^2 - 4ac \in \text{sqr in } \mathbb{Q}$

&

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (\forall x \in \mathbb{Q})$$

T. the quadratic formula over $\mathbb{Z}$

let

- $a, b, c \in \mathbb{Z}$ wh $a \neq 0$
- $x \in$ var $\mathbb{Z}$

then

- $a x^2 + b x + c = 0 \quad (\exists x \in \mathbb{Z})$

$\Leftrightarrow$

$b^2 - 4ac \in$ sqr in $\mathbb{Z}$

&

$b \equiv \pm \sqrt{b^2 - 4ac} \pmod{2a}$

- $ax^2 + bx + c = 0$

$\Leftrightarrow$

$b^2 - 4ac \in$ sqr in $\mathbb{Z}$

&

$$(b \equiv \sqrt{b^2 - 4ac} \pmod{2a}$$

&

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

or

$$b \equiv -\sqrt{b^2 - 4ac} \pmod{2a}$$

&

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}) \quad (\forall x \in \mathbb{Z})$$

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Δ relations between
quadratic polynomials / equations
& their zeros / roots

T. relating the quadratic polynomial & its zeros
to the quadratic equation & its roots

let

- $a, b, c \in \mathbb{C}$ wh $a \neq 0$

- $r =_{df} \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

- $s =_{df} \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

- $x \in \text{var}\,\mathbb{C}$

then

- r and s are the two zeros of the quadratic polynomial

$a x^2 + b x + c$

wiet

r and s are the two roots of the quadratic equation

$a x^2 + b x + c = 0$

wiet

$a x^2 + b x + c = 0$

$\Leftrightarrow$

$x = r \ \vee \ x = s$

$\Leftrightarrow$

$x \in \{r, s\} \quad (\forall x \in \mathbb{C})$

- $a x^2 + b x + c \ = \ a(x - r)(x - s) \quad (\forall x \in \mathbb{C})$

which prescribes an algorithm

for explicitly factoring any quadratic polynomial

over any one of the rings $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$

into two linear factors if it is indeed so factorable

- $r + s = -\dfrac{b}{a} \ \& \ r s = \dfrac{c}{a}$

R. uniqueness of factors

• the two linear factors
of a factorable quadratic polynomial
over one of the rings
$\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$
are essentially unique
inp
they are unique to within a constant factor
and their order as factors

Δ the preceding theorems
are special cases of
more inclusive / general theorems