Bernoulli Numbers
and
Bernoulli Polynomials

#32 of Gottschalk's Gestalts

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of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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□ the Bernoulli numbers
are real numbers
& the Bernoulli polynomials
are polynomials in one real variable
with real coefficients

Δ definition of the Bernoulli numbers

$$B_0,\ B_1,\ B_2,\ \cdots$$

by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (x \in \text{real var};\ |x| < 2\pi)$$

note: evidently B comes from ‘Bernoulli’

Δ recursive definition of the Bernoulli numbers

$$B_0 = 1$$

$$B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k \quad (n \in \text{pos int var})$$

the last equation
may be represented symbolically
in ‘the umbral calculus’ by

$$(B+1)^{n+1} = B_{n+1}$$

wh the LHS is expanded by the BT
& exponents on B are lowered to form subscripts

Δ list of Bernoulli numbers
from index 0 to index 25

$$B_0 = 1$$

$$B_1 = -\frac{1}{2}$$

$$B_2 = \frac{1}{6}$$

$$B_3 = 0$$

$$B_4 = -\frac{1}{30}$$

$$B_5 = 0$$

$$B_6 = \frac{1}{42}$$

$$B_7 = 0$$

$$B_8 = -\frac{1}{30}$$

$$B_9 = 0$$

$$B_{10} = \frac{5}{66}$$

$$B_{11} = 0$$

$$B_{12} = -\frac{691}{2730}$$

$$B_{13} = 0$$

$$B_{14} = \frac{7}{6}$$

$$B_{15} = 0$$

$$B_{16} = -\frac{3617}{510}$$

$$B_{17} = 0$$

$$B_{18} = \frac{43867}{798}$$

$$B_{19} = 0$$

$$B_{20} = -\frac{174611}{330}$$

$$B_{21} = 0$$

$$B_{22} = \frac{854513}{138}$$

$$B_{23} = 0$$

$$B_{24} = -\frac{236364091}{2730}$$

$$B_{25} = 0$$

Δ some basic properties of the Bernoulli numbers

• every Bernoulli number is a rational number

• B_0 is the only Bernoulli number that is a nonzero integer

• every Bernoulli number with plural odd index is zero;
all other Bernoulli numbers are nonzero

• every Bernoulli number
whose index is a positive integer multiple of 4
is a negative rational number;
all other even-indexed Bernoulli numbers
are positive rational numbers

• the nonzero Bernoulli numbers alternate in sign,
starting with $B_0 = 1$

• the absolute values of the even-indexed Bernoulli numbers
attain a minimum value of $\frac{1}{42}$ when the index is 6

• the absolute values of the even-indexed Bernoulli numbers
increase unboundedly and rapidly with increasing index

Δ the Bernoulli numbers appear in many places in mathematics; here are twelve series expansions that use Bernoulli numbers

- $$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} |B_{2n}|\, x^{2n-1} \qquad \left(|x| < \frac{\pi}{2}\right)$$

- $$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}|\, x^{2n-1} \qquad (0 < |x| < \pi)$$

- $$\csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)}{(2n)!} |B_{2n}|\, x^{2n-1} \qquad (0 < |x| < \pi)$$

note: a series expansion for sec x uses the Euler numbers eg

• $\tanh x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n}\, x^{2n-1}$ $\quad \left(|x| < \frac{\pi}{2}\right)$

• $\coth x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n}\, x^{2n-1}$ $\quad (0 < |x| < \pi)$

• $\operatorname{csch} x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)}{(2n)!} B_{2n}\, x^{2n-1}$ $\quad (0 < |x| < \pi)$

note: a series expansion for sech x
uses the Euler numbers eg

- $\log|\sin x| = -\log|\csc x|$

$$= \log|x| - \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n(2n)!} |B_{2n}| \, x^{2n} \qquad (0 < |x| < \pi)$$

- $\log \cos x = -\log \sec x$

$$= -\sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)}{n(2n)!} |B_{2n}| x^{2n} \qquad \left(|x| < \frac{\pi}{2}\right)$$

- $\log|\tan x| = -\log|\cot x|$

$$= \log|x| + \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n-1}-1)}{n(2n)!} |B_{2n}| x^{2n} \qquad \left(0 < |x| < \frac{\pi}{2}\right)$$

- $\log|\sinh x| = -\log|\operatorname{csch} x|$

$$= \log|x| + \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n(2n)!} B_{2n} x^{2n} \qquad (0 < |x| < \pi)$$

- $\log\cosh x = -\log\operatorname{sech} x$

$$= \sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)}{n(2n)!} B_{2n} x^{2n} \qquad \left(|x| < \frac{\pi}{2}\right)$$

- $\log|\tanh x| = -\log|\coth x|$

$$= \log|x| - \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n-1}-1)}{n(2n)!} B_{2n} x^{2n} \qquad \left(0 < |x| < \frac{\pi}{2}\right)$$

Δ the even – indexed Bernoulli numbers
ito the zeta function

B_n

$$= (-1)^{\frac{1}{2}(n+2)} \frac{2n!}{(2\pi)^n} \zeta(n)$$

$$= (-1)^{\frac{1}{2}(n+2)} \frac{2n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{1}{k^n}$$

wh
$n \in$ even pos int

& conversely
zeta of an even positive integer n
ito a Bernoulli number

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \frac{(2\pi)^n}{2n!} |B_n|$$

inp

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots = \frac{\pi^4}{90}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\cdots = \frac{\pi^6}{945}$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\cdots = \frac{\pi^8}{9450}$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\cdots = \frac{\pi^{10}}{93555}$$

Δ definition of the Bernoulli polynomials

$$B_0(x),\ B_1(x),\ B_2(x),\ \cdots \quad (x \in \text{real var})$$

by the generating function

$$\frac{t\,e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (x,\ t \in \text{real var};\ |t| < 2\pi)$$

Δ definition of the Bernoulli polynomials
ito the Bernoulli numbers

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, x^{n-k} \qquad (x \in \text{real var};\ n \in \text{nonneg int})$$

Δ list of Bernoulli polynomials
from degree 0 to degree 10

$$B_0(x) = 1$$

$$B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6}$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$$

$$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$$

$$B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x$$

$$B_8(x) = x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2 - \frac{1}{30}$$

$$B_9(x) = x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x$$

$$B_{10}(x) = x^{10} - 5x^9 + \frac{15}{2}x^8 - 7x^6 + 5x^4 - \frac{3}{2}x^2 + \frac{5}{66}$$

Δ some basic properties of the Bernoulli polynomials
($n \in$ nonneg int; $x \in$ real var)

• every Bernoulli polynomial is monic,
all the coefficients are rational numbers,
the coefficients alternate in sign

• the degree of $B_n(x)$ is n

• $B_n = B_n(0) = (-1)^n B_n(1)$

• $B_n(x+1) - B_n(x) = n\,x^{n-1}$

• $|B_n(x)| \le |B_n| \qquad (0 \le x \le 1 \;\&\; n \in \text{even})$

- $$\frac{d}{dx} B_n(x) = n\, B_{n-1}(x) \qquad (n \geq 1)$$

- $$\int_0^x B_n(t)\, dt = \frac{1}{n+1} \left[B_{n+1}(x) - B_{n+1} \right]$$

- $$\int_x^{x+1} B_n(t)\, dt = x^n$$

- $$\int_0^1 B_n(t)\, dt = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

- $$\int_0^1 B_m(t)\, B_n(t)\, dt = (-1)^{\frac{1}{2}(m+n+2)} \frac{m!\, n!}{(m+n)!} B_{m+n}$$

wh m, n $\in$ pos int

Δ zeta of a plural odd positive integer n
ito a Bernoulli polynomial

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \frac{(2\pi)^n}{2n!} \left| \int_0^1 B_n(t) \cot(\pi t)\, dt \right|$$

Δ sums of powers
of consecutive positive integers
ito
Bernoulli numbers & Bernoulli polynomials

T. let
• $n \in$ pos int var
• $r \in$ nonneg int

then

• $S_r(n)$ wh S comes from 'sum'

= rd siren

= cl the siren polynomial in n
of index r

= df the sum of the rth powers
of the first n positive integers

$$= 1^r + 2^r + 3^r + \cdots + n^r$$

$$= \sum_{k=1}^{n} k^r$$

$$= \frac{1}{r+1}\left[B_{r+1}(n+1) + (-1)^r B_{r+1}\right]$$

$$= \frac{1}{r+1}\left[\sum_{i=0}^{r+1} \binom{r+1}{i} B_i \,(n+1)^{r-i+1} + (-1)^r B_{r+1}\right]$$

$$= \frac{1}{r+1}\sum_{i=0}^{r} (-1)^i \binom{r+1}{i} B_i \, n^{r-i+1}$$

$$= \frac{1}{r+1} n^{r+1} + \frac{1}{2} n^r$$

$$+ \frac{1}{2}\binom{r}{1} B_2\, n^{r-1} + \frac{1}{4}\binom{r}{3} B_4\, n^{r-3} + \frac{1}{6}\binom{r}{5} B_6\, n^{r-5} + \cdots$$

(it is to be understood that

the last sum above has

1 term if $r = 0$

&

$\frac{1}{2}(r+4)$ terms if $r \in$ even ≥ 2

&

$\frac{1}{2}(r+3)$ terms if $r \in$ odd;

there is no constant term;

equivalently the last term contains

either n or n^2)

Δ list of siren polynomials
from degree 1 to degree 12

$$S_0(n) = n$$

$$S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$
$$= \frac{1}{2}n(n+1)$$

$$S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$
$$= \frac{1}{6}n(n+1)(2n+1)$$

$$S_3(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$
$$= \frac{1}{4}n^2(n+1)^2 = \left[\frac{1}{2}n(n+1)\right]^2$$

$$S_4(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

$$S_5(n) = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

$$= \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

$$S_6(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

$$= \frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$$

$$S_7(n) = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$

$$= \frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)$$

$$S_8(n) = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$

$$= \frac{1}{90}n(n+1)(2n+1)(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3)$$

$$S_9(n) = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2$$

$$= \frac{1}{20}n^2(n+1)^2(2n^6 + 6n^5 + n^4 - 8n^3 + n^2 + 6n - 3)$$

$$S_{10}(n) = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

$$= \frac{1}{66}n(n+1)(2n+1)(n^2 + n - 1)$$

$$(3n^6 + 9n^5 + 2n^4 - 11n^3 + 3n^2 + 10n - 5)$$

$$S_{11}(n) = \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2$$

$$= \frac{1}{24}n^2(n+1)^2$$

$$(2n^8 + 8n^7 + 4n^6 - 16n^5 - 5n^4 + 26n^3 - 3n^2 - 20n + 10)$$

T. let

• n, r $\in$ pos int

then

• $S_r(n)$

$$= 1^r + 2^r + 3^r + \cdots + n^r$$

$$= \sum_{k=1}^{n} k^r$$

$$= \int_0^{n+1} B_r(t)\,dt$$

$$= \frac{1}{r+1}\left[B_{r+1}(n+1) - B_{r+1}(0)\right]$$

$$= \frac{1}{r+1}\left[B_{r+1}(n+1) - B_{r+1}\right]$$

T. let

- n $\in$ pos int var
- r $\in$ nonneg int

then

- $S_{r+1}(n) = (r+1)\left[\int_0^n S_r(t)\,dt - n\int_0^1 S_r(t)\,dt\right] + n$

which means that
once you write down

$(r+1)\int_0^n S_r(t)\,dt,$

add n times a coefficient that will make
the sum of the coefficients equal to 1

T. let

- $n \in$ pos int var
- $r \in$ nonneg int

then
referring to the polynomial in n
consisting of the sum
of the nonzero terms in the polynomial
$S_r(n)$

$= 1^r + 2^r + 3^r + \cdots + n^r$

$$= \sum_{k=1}^{n} k^r$$

$$= \frac{1}{r+1} n^{r+1} + \frac{1}{2} n^r$$

$$+ \frac{1}{2}\binom{r}{1} B_2\, n^{r-1} + \frac{1}{4}\binom{r}{3} B_4\, n^{r-3} + \frac{1}{6}\binom{r}{5} B_6\, n^{r-5} + \cdots$$

arranged in canonical order
of decreasing degree

• the degree of the polynomial is $r+1$

• all coefficients of the polynomial are rational numbers

• if $r = 0$, then the polynomial has 1 term

• if $r \in$ even ≥ 2, then the polynomial has $\frac{1}{2}(r+4)$ terms

• if $r \in$ odd, then the polynomial has $\frac{1}{2}(r+3)$ terms

• the leading term of the polynomial is $\frac{1}{r+1}\ n^{r+1}$

• if $r \geq 1$, then

the second term of the polynomial is $\frac{1}{2}\ n^{r}$

• if $r \in$ even, then

the last term of the polynomial is $B_r\ n$

• if $r = 1$, then

the last term of the polynomial is $\frac{1}{2}\ n$

• if $r \in$ odd ≥ 3, then

the last term of the polynomial is $\frac{r}{2} B_{r-1}\ n^2$

• the constant term of the polynomial is 0

ie there is no written constant term

• the first three terms of the polynomial
have degrees that diminish consecutively;
thereafter the degrees of the terms
diminish by two at each step to the right

• the first three terms of the polynomial
have positive coefficients;
thereafter the coefficients of the terms
alternate in sign

• the sum of the coefficients of the polynomial is 1

• if $r \in$ even ≥ 2, then
$n(n+1)(2n+1)$ is a factor of the polynomial

• if $r \in$ odd ≥ 3, then
$n^2(n+1)^2$ is a factor of the polynomial

Δ motivation

¿ why should the generating function
for Bernoulli numbers be considered at all ?
¿ what could be the thoughts
that would lead the mathematician to Bernoulli numbers ?
here are some ideas on the subject;
the mathematician,
roving alone in the universe of mathematics,
is often guided by esthetic principles
to the questions to be considered;
¿ but what is beautiful ?
¿ is not beauty an individual subjective thing ?
¿ is not beauty in the eye of the beholder ?
in mathematics, I would contend,
to some extent yes but not entirely;

perhaps
the most centrally located
transcendental function
in real analysis is
the real exponential function
viz

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (x \in \text{real nr var})$$

(one could say the same thing
about the complex exponential function
in complex analysis);

now transposing 1 from the RHS to the LHS
makes every term on the right contain x as a factor;
that being the case, divide by x;
now we have an analytic function
that is 1 at the origin
(just like the exponential function itself)
and is closely related to the exponential function;
its reciprocal is then analytic
in the neighborhood of the origin
and has a power series in x;
the coefficients are the Bernoulli numbers;
the factorials are just normalizing factors;
compare
the original exponential function
and
the new generating function,
thus:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

¿ what could be prettier ?
¡ voilá !

to attain the Bernouli polynomials,
form the power series in the real variable t
for this generating function

$$\frac{t}{e^t - 1}$$

which gives the Bernoulli numbers;
now form the power series in xt for

$$e^{xt}$$

next multiply together these two series
and collect terms in powers of t,
the coefficients being polynomials in x;
the coefficients are the Bernoulli polynomials;
the factorials are just normalizing factors;
this gives the generating function & expansion
for the Bernoulli polynomials;
when x = 0 we are back to the Bernoulli numbers;

changing the notation a bit
for the purpose of comparison,
we now have
the three elegant equations for
the exponential function,
the Bernoulli numbers,
the Bernoulli polynomials:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

$$\frac{xe^{xt}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}$$

¡ voici !
here they are, all together

the final validation of any mathematical idea
depends on the position it will assume
among the vast body of mathematical entities
and its interactions with them

∆ bioline

James (English)
= Jacques (French)
= Jakob (German)
= Giacomo (Italian)
Bernoulli
1654-1705
Swiss
analyst, combinatorist, geometer,
probabilist, statistician, physicist;
in 1690 he introduced the word ‘integral’;
in 1713 he introduced the Bernoulli numbers
in Ars Conjectandi (Latin) = Art of Conjecturing,
a famous posthumously published work of his
which was the first substantial book
on the theory of probability;
he gave there the formula for the sum of the powers
of the consecutive integers ito the Bernoulli numbers;
he claimed in this book that he was able
by the use of this formula
to calculate the sum of the tenth powers
of the first one thousand positive integers
in less than seven and one-half minutes
and gave the correct sum as
91409924241424243424241924242500
he was a member of the remarkable Bernoulli family
that from the middle 1600’s to the middle 1800’s
produced over a dozen
distinguished mathematicians and physicists GG32-37