Math Snippets: Fourth Bouquet

#31 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
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☐ quantity vs magnitude

the algebraic notion of
quantity = number
eg
integer
rational number
irrational number
(all positive)

and

the geometric notion of
magnitude = measure
eg
length of line/curve
area of region/surface
volume of solid

were virtually inseparable
in the ancient Greek mind

it was not until the Renaissance
say ca 15th century
that it began to creep into
the mathematician's awareness that
algebraic quantity & geometric magnitude
were related but different notions

the independent discovery/invention
of analytic geometry
by Descartes & Fermat ca 1630
was likely
the precipitation point
of clear explicit recognition

of the distinction between
quantity & magnitude

and

of the very basic importance
of the correlation between
quantity & magnitude

□ progresive abstractions/extensions/generalizations
for the notion of
algebraic quantity

• tally numbers
= positive integers

• counting numbers
= positive integers & zero
= nonnegative integers

• whole numbers
= nonnegative integers & negative integers
= integers

• fractions of whole numbers
= quotients of whole numbers
= rational numbers

• measuring numbers
= real numbers
= rational numbers & irrational numbers

• complete ordered field
= the basic axiom system for the real numbers

• complex numbers
= ordered pairs of real numbers

• quaternions
= ordered pairs of complex numbers

• octonions
= ordered pairs of quaternions

• abstract/axiomatic/general algebraic systems
= groups, rings, fields, vector spaces, etc

☐ progressive abstractions/extensions/generalizations
for the notion of
geometric magnitude

• one-dimensional content
in classical euclidean geometry
= length of euclidean line segments
& length of curves in the euclidean plane
or in euclidean 3-space

• two-dimensional content
in classical euclidean geometry
= area of regions in the euclidean plane
& surface area of surfaces in euclidean 3-space

• three-dimensional content
in classical euclidean geometry
= volume of solids in euclidean 3-space

• low-dimensional content
in classical euclidean geometry (in summary)
= curve-length
&
surface-area
&
solid-volume
all in euclidean 3-space

• high-dimensional & any-dimensional content
in euclidean geometry
= contents of structures
in euclidean spaces of arbitrary dimension

• high-dimensional & any-dimensional content
in differential-geometric structures of arbitrary dimension

• measure/integral
in abstract/axiomatic/general
measure/integration theories

□ comments on the Russell paradox

Δ the Russell paradox is briefly derived as follows:

(1) $x =_{df}$ a set variable

(2) $a =_{df} \{x \mid x \notin x\}$

(3) $x \in a \Leftrightarrow x \notin x$

(4) $a \in a \Leftrightarrow a \notin a$ substituting a for x in (3)

? !

contradiction & consternation

Δ the resolution of the paradox

• lines (2) & (3) are equivalent by definition;
now if we read { x | fx } as
'the set of all sets x such that fx'
then we make
the unjustified & incorrect (as it turns out) assumption
that a is a value of x;
go back
& call x a set_0 variable
& read { x | fx } as
'the set_1 of all set_0 x such that fx';
then it is clear that
the passage from (3) to (4) is blocked
because a need not be a value of x
& no contradiction is forced;
indeed
if a is a value of x,
then a contradiction occurs;
hence
a is indeed not a value of x

Δ it is also possible to read { x l fx }
as 'the class of all x such that fx';
this is just a choice of words for convenience:
say 'set, class, etc'
rather than 'set_0, set_1, etc';
everything is still a 'set'
in the loose imprecise natural language of English
but when it comes to precise mathematics
something better is needed

Δ there is an axiom of set theory
called 'the axiom of foundation'
which states that
there is no infinite descending elementhood chain of sets;
as a consequence
no set is an element of itself;
(the idea is that
when a set is formed from certain objects,
something new & different is obtained
& not just one of the original objects)
thus 'all' that Russell's paradox shows
is that
the class of all sets is not a set;
to repeat,
Russell's paradox
is a short proof of the theorem
that the class of all sets is not a set;
every set is a class but not every class is a set

∆ a paradox is an insight gone astray

∆ ¿ just because a word
for a notion in mathematics
is chosen from some natural language such as English
and is a common frequently used word,
why should all of the basic inherent properties
of the notion be immediately clear ?
everything needs definition & clarification
by means of axioms;
even that ultimate notion of set
does not escape this iron-clad demand

∆ bioline
Bertrand Arthur William Russell
1872-1970
English-Welsh
logician, philosopher,
prolific author, controversial public figure;
Russell discovered this paradox in 1902

□ the scientist
the artist
&
the mathematician

• the scientist discovers truth

• the artist creates beauty

• the mathematician,
being both scientist & artist,
does both

• ¿ can it be said that
the mathematician
discovers/creates
truth/beauty
in all $2 \times 2 = 4$ possible combinations ?

□ scientific revolutions include:

- Big Bang cosmology
- computers
- evolution
- molecular biology
- plate tectonics
- quantum mechanics
- relativity

¿ what parts of mathematics
should be classified as
scientific revolutions ?

¿ none/some/all of mathematics ?

☐ the seven pillars of classical analysis

∆ the two giants of the 17th century

• Newton
1642-1727
English
• Leibniz
1646-1716
German

∆ the two giants of the 18th century

• Euler
1707-1783
Swiss
• Lagrange
1736-1813
French

∆ the three giants of the 19th century

• Cauchy
1789-1857
French
• Weierstrass
1815-1897
German
• Riemann
1826-1866
German

☐ a line of Shakespeare
subjected to an attempt
to translate it
from English into Mathematics

$2B \vee \neg 2Bie$?

• this string of symbols is not mathematics;
this string is nonsense;
this string has syntactic coherence
but not semantic coherence
in that for significance
it relies on readings of individual symbols
and on homophones,
and it at least partially ignores
their individual & composite meanings;
this string provides a lesson about mathematics
but is not mathematics itself

• ¿ what is the lesson ?

• the lesson is that
mathematics makes use of symbolism
but is not symbolism itself

• ¿ does this translation suggest that Shakespeare
was prophesying the intuitionist challenge
to the law of the excluded middle ?

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□ two simpler suggestive symbolic forms
of more complicated precise detailed formulas
thanks to
the binomial theorem
& ' the umbral calculus'

• the symbolic form of the formula
for the nth ($n \in$ pos int) derivative
of the product of two real functions u & v
of a single variable

$$(u\,v)^{(n)} = \left(D\,u + D\,v \right)^n$$

• the symbolic form of the formula
to recursively determine
the Bernoulli numbers

$$B_0 = 1$$

$$\left(B + 1 \right)^n = B_n \quad (n \in \text{int} \geq 2)$$

□ ¿ simplest transcendental number ?

• just as

$\sqrt{2}$ = root two
is the simplest irrational number

so

$2^{\sqrt{2}}$ = two to root two
is the simplest transcendental number

• ¿ what does 'simplest' mean in this context ?

¿ 'simplest' to
define
describe
look at
prove
remember
state ?

• in 1844 Liouville showed that the number

$$\sum_{n=1}^{\infty} 10^{-n!}$$

(viz there are 1's in the n! decimal places & 0's elsewhere)
and infinitely many others are transcendental;
this was the first existence proof
of transcendental numbers and it was constructive

• Joseph Liouville
1809-1882
French
analyst, geometer, number theorist, applied mathematician,
founder and editor of a mathematical research journal

• in 1872 Cantor gave a nonconstructive(?) existence proof of transcendental numbers;
indeed he showed that
uncountably many transcendental numbers exist

• Georg Ferdinand Ludwig Philipp Cantor
1845-1918
German
analyst, set theorist;
the primary founder of set theory

• among the famous 23 problems
posed by Hilbert in 1900 in Paris
the 7th was the problem of
proving the transcendence of certain numbers
eg
$2^{\sqrt{2}}$

• David Hilbert
1862-1943
German
algebraist, analyst, geometer, logician,
applied mathematician, philosopher;
founder of formalist school of mathematics;
said to be the last universal mathematician

• in 1934 Gelfond proved
the Hilbert number $2^{\sqrt{2}}$ is transcendental
as a corollary to
a more general theorem of his

• Aleksandr Osipovich Gelfond
1906-1968
Russian
analyst, number theorist

• ca 300 BCE Euclid proved
in his famous book 'The Elements'
that $\sqrt{2}$ is irrational

• Euclid of Alexandria
ca 365 - ca 300 BCE
Greek
geometer, number theorist;
the most prominent mathematician of antiquity
& the leading mathematics teacher of all time
because
his great inclusive compilation
& axiomatic organization
of the mathematics of his time,
'The Elements',
has continued to be substantially used
as a textbook for two millenia;
the axiomatic method in mathematics
was first fully established
in 'The Elements'

□ a slogan for differential geometry

• differentiate & interpret

• in more detail
this is a procedure
for research
in differential geometry:

(1) isolate the object to be studied

(2) express the object in a coordinate system

(3) differentiate the expression

(4) interpret the result

• a good elementary example
of this procedure
are the Frenet-Serret formulas

☐ statistical gradations in increasing specialization

agent subject matter

- mathematician .... pure mathematics
- analyst .... pure mathematics
- measure theorist .... pure mathematics
- probabilist .... pure mathematics
- mathematical statistician .... applications of mathematics
- applied statistician .... sciences & other fields
- statistics user .... sciences & other fields

where
an applied statistician is
an expert in statistics as applied to many fields
& a statistics user is
a substantive scientist
or actuary
or agent
of business, government, industry, insurance, etc

note: statistics is a science & uses much mathematics
but is not itself a branch of mathematics GG31-23

□ T for two

$T_1 =_{df} 2$

$T_2 =_{df} 2^{T_1}$

$T_3 =_{df} 2^{T_2}$

$\vdots$

$T_{n+1} =_{df} 2^{T_n}$ $\quad$ ($n \in$ pos int)

• this is an example of
'stacked exponential growth';
note that n is the number of 'stories'
in the 'high rise' T_n;
think of T as standing for 'tall' and 'tower'

• Hilbert was fond of saying that the determination of
the middle digit or digits of T_n ($n \geq 10$),
although finitely attainable in principle,
was beyond present human capability;
once in his lectures he inadvertently replaced
the base 2 by the base 10

□ there are two complementary procedures
in the discovery/invention & learning/teaching
of mathematics
viz

• from the bottom up:
first study various special objects
& then
eventually write down
a definition = an axiom system
that includes all the special objects
& study the possibly more general notion
determined by the axiom system

• from the top down:
first write down
a definition = an axiom system
& then
study the general notion
determined by the axiom system
(as well as various special cases)

note:
it is not constructive/instructive to ask
¿ which procedure is better ?
both are essential
for the acquisition/development/production
of mathematics

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D. the ubiquitous constant

$=_{ab}$ ubiq const

$=_{dn}$ U wh U $\leftarrow$ <u>u</u>biquitous

$$=_{df} \text{AGM}\left(1, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{2\sqrt{2}} \div \int_0^1 \frac{1}{\sqrt{(1-t^2)(2-t^2)}}\,dt$$

$$= \frac{\pi}{2\sqrt{2}} \div \int_0^{\pi/2} \frac{1}{\sqrt{2-\sin^2\theta}}\,d\theta$$

$$= \frac{\Gamma^2(3/4)}{\sqrt{\pi}}$$

$$= 0.84721\ 30848\cdots$$

☐ English
is recognized as
the world-wide language for:

business
communication
computers
diplomacy
flight
medicine
science
sports
technology
tourism

English
is also
the best language for
mathematics
IMHO

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□ ¿ why formalize ?
¡ just imagine it but don't do it !

evidence is strong that all 'ordinary mathematics'
could be paraphrased completely within a formal system
consisting of an axiomatic set theory
with the lower predicate calculus as logical vehicle;
in principle, such a program could actually be carried out
and written down for all to look at;
interestingly enuf, the result would be catastrophic;
great losses would occur:
loss of accessibility
loss of brevity
loss of clarity
loss of comprehension
loss of insight
loss of motivation
loss of time
etc;
there would be questionable gain of precision and security;
it is doubtful whether there would be
more assurance of freedom from errors & contradictions;
formal systems
are
great instruments of investigation for certain topics
but
they are not the way
the mathematician
thinks & understands & records & communicates

☐ names/readings of the number/numeral 0 in English
additive identity (element)
aught
cipher
love (as a score in sports eg tennis;
possibly from French l'oeuf = the egg)
naught
nil
none
nothing
null
nullity
oh
zero
• in American English slang
beans
big oh
diddly squat
goose egg
nix
zilch
zip
zippo
• 'nothing' in other languages
nada (Spanish)
nichts (German)
niente (Italian)
nihil (Latin)
rien (French)