Math Snippets: Second Bouquet

#29 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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GG29-2

- ☐ facts/data/information/knowledge/science

facts	collected are
↓	
data	arranged are
↓	
information	organized is
↓	
knowledge	explained is
↓	
science	

D. floor & ceiling

let

$x \in$ real nr

then

• the floor of x

$=_{dn} \lfloor x \rfloor$

$=_{rd}$ floor (of) x

$=_{df}$ the greatest integer n st $n \le x$

wh

the two - part sign $\lfloor \cdots \rfloor$

$=_{cl}$ the floor sign = the el brackets

• the ceiling of x

$=_{dn} \lceil x \rceil$

$=_{rd}$ ceiling (of) x

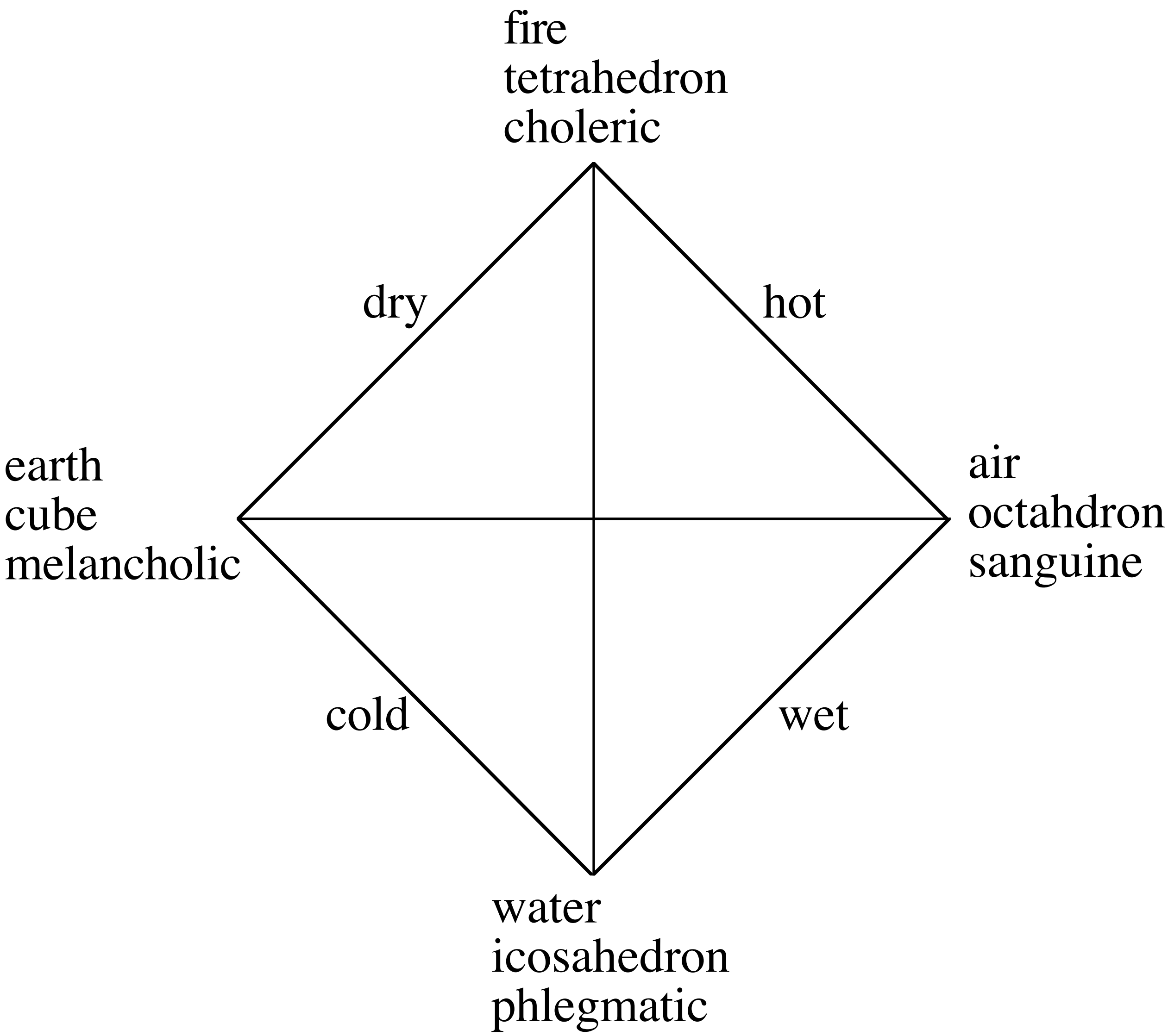
$=_{df}$ the least integer n st $x \le n$

wh

the two - part sign $\lceil \cdots \rceil$

$=_{cl}$ the ceiling sign = the gamma brackets

☐ the cosmic scheme of the four elements from long ago



modern science relates in the four states of matter:
earth = solid
water = liquid
air = gas
fire = plasma;
Plato said the regular dodecahedron of the five regular solids corresponds to the Universe

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☐ Gabriel's Horn: the painter's paradox

• consider
the unbounded region R with area A
under the equilateral hyperbola $y = 1/x$
&
above the x-axis from $x = 1$ to $x = +\infty$;
revolve the region R about the x-axis to obtain
an unbounded solid (call it 'Gabriel's Horn')
with volume V & surface area S ;
then it follows that
A is infinite
and
S is infinite
but
V is finite !

• thus in this case
it takes an infinite amount of paint
to cover a solid with finite volume

• Gabriel's Horn can be filled up but cannot be painted

□ a general guide to point the way
= a rough rule of thumb:
analysis via geometric/physical insight/intuition

• 'integrate a little piece to find the whole thing'

• to find a geometric/physical quantity,
determine its
differential = 'element of quantity'
& then integrate;
in symbols

$$Q = \int dQ$$

and in words

'quantity equals integral of element of quantity'

• this dictum
is to be regarded as
a shorthand suggestion for
a more complicated limit process
in any given instance

☐ the mathematical gospel according to Saint Hardy

∆ Hardy says that a mathematician
is
'a maker of patterns of ideas'

∆ Hardy's five components of mathematical beauty are:
• generality
• depth
• unexpectedness
• inevitability
• economy
where their initial-letter mnemonic is GUIDE

∆ Hardy's 'pure-talent score' of mathematicians
included the valuations:

• Hardy	=	25
• Littlewood	=	30
• Hilbert	=	80
• Ramanujan	=	100

∆ bioline
Godfrey Harold Hardy
1877-1947
English
analyst, number theorist;
cricket aficionado,
first established mathematician
to recognize the genius of Ramanujan
GG29-8

□ Hilbert's Hotel

= a hotel which has aleph-null rooms

to accommodate aleph-null guests

& therefore

no one will ever be turned away

because they lack a reservation

even tho the hotel may be full

□ historical numbers

remember numerical approximations
by correlating them with historical events
& giving them associated names;
here are some examples

• the Columbus number
= the square root of 2
= $\sqrt{2}$
= $1.4142+$
because Columbus discovered America in 1492
&
4+1+4 = 9
besides the evident other correspondences

• the first presidential number
= the George Washington number
= the square root of 3
= $\sqrt{3}$
= $1.732+$
because George Washington,
the first president of the United States,
was born in the year 1732

• the seventh presidential number
= the Andrew Jackson number
= the natlogbase
= e
= 2.718281828459045+
because Andrew Jackson was
the 7th president of the United States,
he served 2 terms,
and he was first elected in 1828;
that accounts for the integer
& for the first nine decimal places in e;
the last six decimal places given above are
the angles in degrees
of the three angles
of an isosceles right triangle

□ the first way
is
the hardest way

• it often happens that
once a theorem has been proved,
proofs much easier and more insightful than the first
are found

• it sometimes happens that
the first 'proof' of a theorem
is actually invalid
because
for example
it depends on notions that were not yet
clearly defined and developed
at the moment the 'proof' was offered;
in such a case
the pioneer mathematician
is ahead of their time;
insight/intuition leads the way,
rigor follows

□ notation for infinite numbers

• cardinal numbers

$=_{ab}$ cardinals

$\aleph_0$ = the least infinite cardinal = 'aleph / alef null'

$(\aleph_\alpha \mid \alpha \in \mathrm{Ord})$ = the ladder of infinite cardinals

c = the cardinal of the continuum = 'little cee'

Crd = the class of all cardinals

• ordinal numbers

$=_{ab}$ ordinals

ω = the least infinite ordinal = 'little omega'

$(\omega_\alpha \mid \alpha \in \mathrm{Ord})$ = the ladder of infinite initial ordinals

Ω = the least uncountable ordinal = 'big omega'

Ord = the class of all ordinals

□ mathematics
&
mathematics exposition

to simplify
&
to unify
mathematics
continually

requires mathematicians

to simplify
&
to unify
mathematics exposition
continually

☐ a geometric/kinematic insight into the nature of a curve

• a smooth curve may be considered to be the path
which is simultaneously
traced by a moving point
&
enveloped by a moving line
where the point is on the line

• the curve is produced by action of the point-line pair,
the point being on the curve
&
the line being tangent to the curve at the point

• the point moves continuously on the curve
&
the line moves continuously about the point

• the line moves continuously touching the curve
&
the point moves continuously on the line

• the motion of the point determines the motion of the line
&
the motion of the line determines the motion of the point

• the position of the point determines the position of the line
&
the position of the line determines the position of the point

• the tangent line to the curve
at any given point of the curve
is the linear = first-degree approximation to the curve
in the neighborhood of the given point
&
the tangent line is a one-dimensional
linear space = vector space
in its own right
with origin at the given point

☐ a geometric/kinematic insight into the nature of a surface

• a smooth surface may be considered to be the spread which is simultaneously
traced by a moving point
&
enveloped by a moving plane
where
the point is on the plane

• the surface is produced by action of the point-plane pair,
the point being on the surface
&
the plane being tangent to the surface at the point

• the point moves continuously on the surface
&
the plane moves continuously about the point

• the plane moves continuously touching the surface
&
the point moves continuously on the plane

• the motion of the point determines the motion of the plane
&
the motion of the plane determines the motion of the point

• the position of the point determines the position of the plane
&
the position of the plane determines the position of the point

• the tangent plane to the surface
at any given point of the surface
is the linear = first-degree approximation to the surface
in the neighborhood of the given point
&
the tangent plane is a two-dimensional
linear space = vector space
in its own right
with origin at the given point

☐ the shortest known proof of
the Pythagorean theorem

• the altitude to the hypotenuse of a right triangle
divides the triangle
into two smaller triangles similar to the original triangle;
the sum of the areas of the two smaller triangles
equals the area of the original triangle;
the areas of all three triangles
are proportional to
the squares of their hypotenuses
with the same constant of proportionality
because the triangles are similar;
substitute in the equation
and divide by the constant

• this proof is a good example of
a proof that makes the theorem obvious
(once you see the proof, of course)

• this proof is a good example of
a mathematical gestalt:
an immediate clear comprehensive structured unified
image of a significant amount of mathematics

• this is perhaps
the most insightful proof known
of the Pythagorean theorem
& it's all in words
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☐ a quick mental calculation

- 2^{20}

$$= \left(2^{10}\right)^2$$

$$= (1024)^2$$

$$= (1000 + 24)^2$$

$$= 1000^2 + 2 \times 1000 \times 24 + 24^2$$

$$\begin{aligned} = {} & 1\,000\,000 \\ & +48\,000 \\ & +576 \end{aligned}$$

$$= 1\,048\,576$$

□ mathematics vs computer science

• what theorems are to mathematics,
algorithms are to computer science

• the above statement may be paraphrased
in the form of
a semantic proportion
= an equality of ratios of notions
viz

theorems : mathematics = algorithms : computer science

read
theorems are to mathematics
as
algorithms are to computer science

☐ a maximum maxim
for
mathematics

algebraic formulas

and

geometric forms

merge to make

magnificent mathematics

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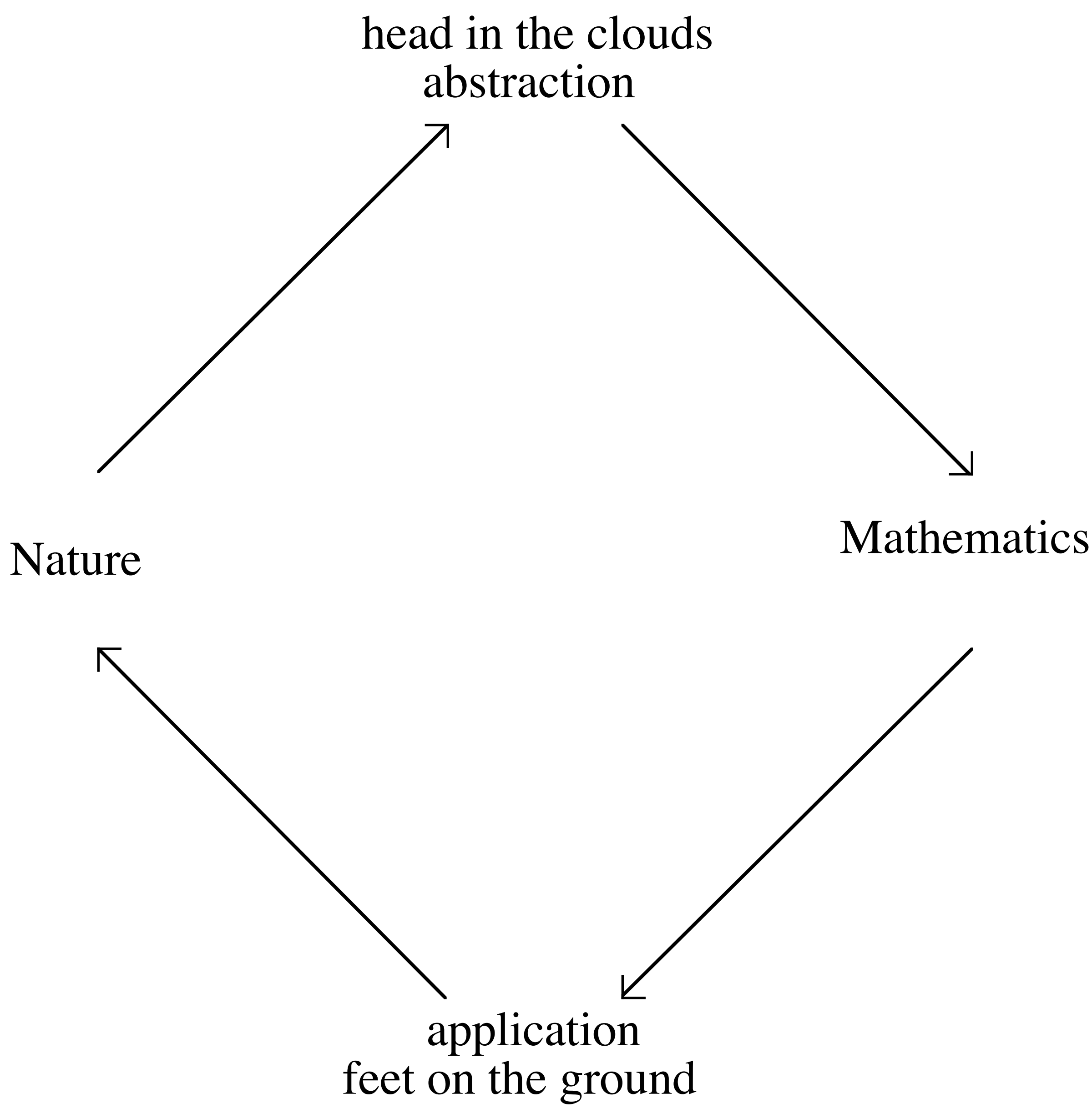
□ natural language vs formal language

• treat a natural language
such as English
as a rigorous logical instrument
ie
as a formal language/system
&
as a result you can get
an inference such as this one:

science is knowledge
and
knowledge is power
and
power corrupts;
therefore
science corrupts

¿ what happened ?
altho a natural language
has a lot of correct logic
built into it,
yet the individual words
and the grammar
are so ambiguous
that logical errors are inevitable

□ the Nature/Mathematics cycle



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□ some notable sets of integers

- the positive integers

= $\mathbb{P}$

- the even positive integers

= $2\mathbb{P}$

- the odd positive integers

= $2\mathbb{P} - 1$

- the nonnegative integers

= $\mathbb{N}$

- the even nonnegative integers

= $2\mathbb{N}$

- the odd nonnegative integers

= the odd positive integers

= $2\mathbb{N} + 1 \;=\; 2\mathbb{P} - 1$

- the negative integers

$= -\mathbb{P}$

- the even negative integers

$= -2\mathbb{P}$

- the odd negative integers

$= -2\mathbb{P} + 1$

- the nonpositive integers

$= -\mathbb{N}$

- the even nonpositive integers

$= \ -2\mathbb{N}$

- the odd nonpositive integers

= the odd negative integers

$= \ -2\mathbb{N} - 1 = \ -2\mathbb{P} + 1$

- the integers

$= \mathbb{Z}$

- the even integers

$= 2\mathbb{Z}$

- the odd integers

$= 2\mathbb{Z}+1$

- the n - multiples $(n \in \text{int})$

= the additive group of n - multiples

= $n\mathbb{Z}$

- the n - multiples plus k $(n, k \in \text{int})$

= the equivalence class of integers modulo n that contains k $(n \in \text{pos int})$

= $n\mathbb{Z} + k$

- the additive group of all equivalence classes of integers modulo n $(n \in \text{pos int})$

= the additive group of integers mod n

= $\mathbb{Z} / n\mathbb{Z}$

$=_{dn} \mathbb{Z}_n$

□ extremely special sets

Δ the largest of all sets

under consideration at the moment iie

$=_{cl}$ the universal set

= the universe of discourse (in an older terminology)

= the domain of individuals (in a newer terminology)

= the space

$=_{dn}$ V

$=_{rd}$ (cap) vee

[where the letter V may be thought of as

an ancient / stylized capital letter yu

for the initial letter of the word 'universe']

$=_{dn}$ S̸

$=_{rd}$ (crossed cap) ess

[where the letter S is

the capitalized initial letter of the word 'space';

note that

ess S is to crossed ess S̸

as

oh O is to crossed oh ∅]

Δ the smallest of all sets

$=_{cl}$ the empty / null / vacuous / void set

$=_{dn}$ Λ

$=_{rd}$ (cap) lambda

[where the letter Λ is
the capital Greek letter lambda
which is the capitalized initial letter of the Greek noun
λακκος = gap, hole, pit, void & also an inverted V]

$=_{dn}$ $\varnothing$

$=_{rd}$ (crossed) oh

[where the letter crossed oh ∅
comes from the letter oh O which is like the numeral zero 0
which is the number of the elements of the empty set ∅;
the symbol ∅
is a Scandinavian (Danish & Norwegian) vowel letter
&
also a vowel symbol of the International Phonetic Alphabet
with sound as in
French feu = fire & German schön = beautiful;
there is no equivalent sound in English]

Δ note the duality:

(1) tfsape

- A $= \varnothing$
- A $= \Lambda$
- A is empty
- A is the empty set
- for no x, $x \in A$
- for all x, $x \notin A$

(2) tfsape

- A = $
- A $= V$
- A is spacial
- A is the space
- for all x, $x \in A$
- for no x, $x \notin A$

Δ the extreme sets

$=_{df}$ the two sets

$\varnothing = \Lambda$ & $ = V