# A Semi-Systematic Sampling of Trig Identities: Part IV

#26 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms of the Organization & Exposition of Mathematics

by Walter Gottschalk

Infinite Vistas Press
PVD RI
2001

GG26-1 (43)

500 Angell St #414
Providence RI 02906


GG26-2

$\Delta$ sin of 3 - angle sum ito sin & cos of summands

- $\sin(A+B+C)$

$= \sin A \cos B \cos C$

$+ \cos A \sin B \cos C$

$+ \cos A \cos B \sin C$

$- \sin A \sin B \sin C$

$$= 2\cos A \cos B \cos C - \begin{vmatrix} \sin A & \cos B & \cos C \\ \cos A & \sin B & \cos C \\ \cos A & \cos B & \sin C \end{vmatrix}$$

$\Delta$ cos of 3 - angle sum ito sin & cos of summands

• $\cos(A+B+C)$

$= \cos A \cos B \cos C$

$- \cos A \sin B \sin C$

$- \sin A \cos B \sin C$

$- \sin A \sin B \cos C$

$$= \begin{vmatrix} \cos A & \sin B & \sin C \\ \sin A & \cos B & \sin C \\ \sin A & \sin B & \cos C \end{vmatrix} - 2 \sin A \sin B \sin C$$

$\Delta$ looking at the two preceding formulas for

$\sin(A+B+C)$ & $\cos(A+B+C)$,

to pass from one to the other

interchange sin & cos

and change the sign of the RHS

$\Delta$ tan of 3 - angle sum ito tan of summands

- $\tan(A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}$$

wh

$$\text{numerator} = 2 - \begin{vmatrix} \tan A & 1 & 1 \\ 1 & \tan B & 1 \\ 1 & 1 & \tan C \end{vmatrix}$$

&

$$\text{denominator} = \begin{vmatrix} 1 & \tan B & \tan C \\ \tan A & 1 & \tan C \\ \tan A & \tan B & 1 \end{vmatrix} - 2 \tan A \tan B \tan C$$

$\Delta$ cot of 3 - angle sum ito cot of summands

• $\cot(A+B+C)$

$$= \frac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1 - \cot A \cot B - \cot A \cot C - \cot B \cot C}$$

wh

$$\text{numerator} = 2 - \begin{vmatrix} \cot A & 1 & 1 \\ 1 & \cot B & 1 \\ 1 & 1 & \cot C \end{vmatrix}$$

&

$$\text{denominator} = \begin{vmatrix} 1 & \cot B & \cot C \\ \cot A & 1 & \cot C \\ \cot A & \cot B & 1 \end{vmatrix} - 2 \cot A \cot B \cot C$$

$\Delta$ looking at the two preceding formulas for
$\tan(A+B+C)$ & $\cot(A+B+C)$,
to pass from one to the other
interchange tan & cot

Δ to write down the formula for
the tangent of the sum of finitely many angles
ito the tangents of the individual angles,
form a quotient in which

- the numerator
is the alternating sum
of the odd - numbered symmetric functions
of the tangents of the individual angles
&
- the denomnator
is the alternating sum
of the even - numbered symmetric functions
of the tangents of the individual angles

Δ a somewhat similar rule holds for the cotangent
but depends on the parity of the number of angles

Δ product of sin & cos of 3 angles as sum

• $4 \sin A \sin B \sin C$

$= \ -\sin(A+B+C)$

$+\sin(-A+B+C)$

$+\sin(A-B+C)$

$+\sin(A+B-C)$

- $4 \sin A \sin B \cos C$

$$= \quad -\cos(A+B+C)$$

$$+\cos(-A+B+C)$$

$$+\cos(A-B+C)$$

$$-\cos(A+B-C)$$

- $4 \sin A \cos B \sin C$

$= \quad -\cos(A+B+C)$

$+\cos(-A+B+C)$

$-\cos(A-B+C)$

$+\cos(A+B-C)$

- $4\sin A\cos B\cos C$

$= \sin(A+B+C)$

$-\sin(-A+B+C)$

$+\sin(A-B+C)$

$+\sin(A+B-C)$

- $4 \cos A \sin B \sin C$

$= \quad -\cos(A+B+C)$

$-\cos(-A+B+C)$

$+\cos(A-B+C)$

$+\cos(A+B-C)$

- $4\cos A \sin B \cos C$

$= \sin(A+B+C)$

$+\sin(-A+B+C)$

$-\sin(A-B+C)$

$+\sin(A+B-C)$

- $4 \cos A \cos B \sin C$

$= \sin(A + B + C)$

$+ \sin(-A + B + C)$

$+ \sin(A - B + C)$

$- \sin(A + B - C)$

- $4\cos A\cos B\cos C$

$$= \cos(A+B+C)$$

$$+\cos(-A+B+C)$$

$$+\cos(A-B+C)$$

$$+\cos(A+B-C)$$

$\Delta$ some three - angle identities

- $\sin A + \sin B + \sin C$

$$= \sin(A+B+C) + 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$$

- $\cos A + \cos B + \cos C$

$$= -\cos(A+B+C) + 4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}$$

- $\tan A + \tan B + \tan C$

$$= \tan A\tan B\tan C + \frac{\sin(A+B+C)}{\cos A\cos B\cos C}$$

- $\cot A + \cot B + \cot C$

$$= \cot A\cot B\cot C - \frac{\cos(A+B+C)}{\sin A\sin B\sin C}$$

- $\sin(A-B)+\sin(B-C)+\sin(C-A)$

$$= -4\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}$$

- $\cos(A-B)+\cos(B-C)+\cos(C-A)$

$$= 4\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}$$

- $\sin^2(A+B+C)$

$+\sin^2(-A+B+C)$

$+\sin^2(A-B+C)$

$+\sin^2(A+B-C)$

$= 2-2\cos 2A\cos 2B\cos 2C$

- $\cos^2(A+B+C)$

$+\cos^2(-A+B+C)$

$+\cos^2(A-B+C)$

$+\cos^2(A+B-C)$

$= 2+2\cos 2A\cos 2B\cos 2C$

- $\sin 4A + \sin 4B + \sin 4C$

$= 4\sin 2A \sin 2B \sin 2C$

$+ 2\sin 2(A + B + C)$

$+ 8\cos(A + B + C)$

$\times \sin(-A + B + C)$

$\times \sin(A - B + C)$

$\times \sin(A + B - C)$

$\Delta$ some trig eqns that are true
for all angles A, B, C
such that their sum is a straight angle:
$A + B + C = 180^{o} = \pi^{r}$;
hence these formulas hold
for the angles A, B, C of any triangle

- $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

- $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

- $\sin A - \sin B + \sin C = 4\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

- $-\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

- $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

- $\cos A + \cos B - \cos C = 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} - 1$

- $\cos A - \cos B + \cos C = 4\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} - 1$

- $-\cos A + \cos B + \cos C = 4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 1$

- $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

- $\sin 3A + \sin 3B + \sin 3C + 4 \cos \frac{3}{2} A \cos \frac{3}{2} B \cos \frac{3}{2} C = 0$

- $\sin 4A + \sin 4B + \sin 4C + 4 \sin 2A \sin 2B \sin 2C = 0$

- $\cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C + 1 = 0$

- $\cos 3A + \cos 3B + \cos 3C + 4 \sin \frac{3}{2} A \sin \frac{3}{2} B \sin \frac{3}{2} C = 0$

- $\cos 4A + \cos 4B + \cos 4C + 1 = 4 \cos 2A \cos 2B \cos 2C$

• $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$

$= 1 + 4\sin\frac{1}{4}(A+B)\sin\frac{1}{4}(B+C)\sin\frac{1}{4}(C+A)$

• $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$

$= 4\cos\frac{1}{4}(A+B)\cos\frac{1}{4}(B+C)\cos\frac{1}{4}(C+A)$

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

- $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

- $(\cot A + \cot B + \cot C)^2 = \cot^2 A + \cot^2 B + \cot^2 C + 2$

- $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

- $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$

- $\left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right)^2 = \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} + 2$

- $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$

- $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$
- $\sin^2 A - \sin^2 B + \sin^2 C = 2\sin A \cos B \sin C$
- $-\sin^2 A + \sin^2 B + \sin^2 C = 2\cos A \sin B \sin C$

- $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

- $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C$
- $\cos^2 A - \cos^2 B + \cos^2 C = 1 - 2\sin A \cos B \sin C$
- $-\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \sin B \sin C$

- $\dfrac{\sin 2A}{\tan B + \tan C} = \dfrac{\sin 2B}{\tan C + \tan A} = \dfrac{\sin 2C}{\tan A + \tan B}$

$= 2\cos A \cos B \cos C$

- $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \cot \dfrac{C}{2}$ & cyclicly

Metatheorem.

• the formula $F(A, B, C)$ is valid

$\Rightarrow$

• the formula $F\left(A-\pi, B+\frac{\pi}{2}, C+\frac{\pi}{2}\right)$ is valid

& cyclicly

• the formula $F(-A, \pi-B, \pi-C)$ is valid

& cyclicly

• the formula $F(A+\pi, B+\pi, C-2\pi)$ is valid

• the formula $F(\pi-2A, \pi-2B, \pi-2C)$ is valid

• the formula $F(3A, 3B-\pi, 3C-\pi)$ is valid

& cyclicly

• the formula $F\left(\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}, \frac{\pi}{2}-\frac{C}{2}\right)$ is valid

• etc

T. let

$A,\ B,\ C\ \in$ the angles of a triangle ABC

then

(1) considering $\sin A + \sin B + \sin C$

as a function of the triangle ABC:

• $0 < \sin A + \sin B + \sin C \leq \frac{3}{2}\sqrt{3}$

• $\sin A + \sin B + \sin C\ =\ \frac{3}{2}\sqrt{3}$

$\Leftrightarrow$ $\ ABC \in$ eq $\Delta$

• $\sin A + \sin B + \sin C$ assumes all values

in the left - open right - closed real interval

from 0 to $\frac{3}{2}\sqrt{3}$

(2) considering $\sin 2A + \sin 2B + \sin 2C$

as a function of the triangle ABC:

• $0 < \sin 2A + \sin 2B + \sin 2C \leq \frac{3}{2}\sqrt{3}$

• $\sin 2A + \sin 2B + \sin 2C = \frac{3}{2}\sqrt{3}$

$\Leftrightarrow$ ABC $\in$ eq $\Delta$

• $\sin 2A + \sin 2B + \sin 2C$ assumes all values in the left - open right - closed real interval from 0 to $\frac{3}{2}\sqrt{3}$

(3) considering $\sin 3A + \sin 3B + \sin 3C$
as a function of the triangle ABC:

• $-2 < \sin 3A + \sin 3B + \sin 3C \le \frac{3}{2}\sqrt{3}$

• $\sin 3A + \sin 3B + \sin 3C = 0$
$\Leftrightarrow$ $A = 60^{o}$ or $B = 60^{o}$ or $C = 60^{o}$

• $\sin 3A + \sin 3B + \sin 3C = \frac{3}{2}\sqrt{3}$
$\Leftrightarrow$ $ABC \in$ isos $\Delta$ with apex angle $140^{o}$

• $\sin 3A + \sin 3B + \sin 3C$ assumes all values
in the left - open right - closed real interval
from $-2$ to $\frac{3}{2}\sqrt{3}$

(4) considering $\sin 4A + \sin 4B + \sin 4C$

as a function of the triangle ABC:

- $-\frac{3}{2}\sqrt{3} \;\le\; \sin 4A + \sin 4B + \sin 4C \;\le \frac{3}{2}\sqrt{3}$

- $\sin 4A + \sin 4B + \sin 4C \;=\; \frac{3}{2}\sqrt{3}$

$\Leftrightarrow$ ABC $\in$ isos $\Delta$ with apex angle $120^{\text{o}}$

- $\sin 4A + \sin 4B + \sin 4C \;=\; 0$

$\Leftrightarrow$ ABC $\in$ rt $\Delta$

- $\sin 4A + \sin 4B + \sin 4C \;= -\frac{3}{2}\sqrt{3}$

$\Leftrightarrow$ ABC $\in$ eq $\Delta$

- $\sin 4A + \sin 4B + \sin 4C$ assumes all values

in the closed real interval

from $-\frac{3}{2}\sqrt{3}$ to $\frac{3}{2}\sqrt{3}$

□ Chebyshev polynomials

Δ the Chebyshev polynomials
of the first kind
$=_{ab}$ CP1
$=_{dn}$ $T_n(x)$ ($n \in$ nonneg int var; $x \in$ real var)
$=_{dr}$

- $T_0(x) = 1$
- $T_1(x) = x$
- $T_{n+2}(x) = 2\,x\,T_{n+1}(x) - T_n(x)$

Δ the Chebyshev polynomials
of the second kind
$=_{ab}$ CP2
$=_{dn}$ $U_n(x)$ ($n \in$ nonneg int var; $x \in$ real var)
$=_{dr}$

- $U_0(x) = 1$
- $U_1(x) = 2\,x$
- $U_{n+2}(x) = 2\,x\,U_{n+1}(x) - U_n(x)$

Δ relations between CP1 & CP2

- $T_{n+1}(x) = U_{n+1}(x) - x\,U_n(x)$   (n ∈ nonneg int)

- $U_{n+1}(x) = \dfrac{x\,T_n(x) - T_{n+1}(x)}{1 - x^2}$   (n ∈ nonneg int)

$\Delta$ the general multiple - angle formulas
for the cosine & sine
ito the Chebyshev polynomials

- $\cos n A = T_n (\cos A)$ wh $n \in$ pos int
- $\cos nA = (-1)^{\frac{n}{2}} T_n(\sin A)$ wh $n \in$ even pos int
- $\cos nA = (-1)^{\frac{n-1}{2}} \cos A \, U_{n-1}(\sin A)$ wh $n \in$ odd pos int

- $\sin n A = \sin A \, U_{n-1}(\cos A)$ wh $n \in$ pos int
- $\sin nA = (-1)^{\frac{n}{2}-1} \cos A \, U_{n-1}(\sin A)$ wh $n \in$ even pos int
- $\sin nA = (-1)^{\frac{n-1}{2}} T_n(\sin A)$ wh $n \in$ odd pos int

Δ the Chebyshev polynomials of the first kind
listed explicitly for degrees up to 10

- $T_0(x) = 1$
- $T_1(x) = x$
- $T_2(x) = 2x^2 - 1$
- $T_3(x) = 4x^3 - 3x$
- $T_4(x) = 8x^4 - 8x^2 + 1$
- $T_5(x) = 16x^5 - 20x^3 + 5x$
- $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
- $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$
- $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
- $T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$
- $T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$

$\Delta$ the Chebyshev polynomials of the second kind listed explicitly for degrees up to 10

- $U_0(x) = 1$
- $U_1(x) = 2x$
- $U_2(x) = 4x^2 - 1$
- $U_3(x) = 8x^3 - 4x$
- $U_4(x) = 16x^4 - 12x^2 + 1$
- $U_5(x) = 32x^5 - 32x^3 + 6x$
- $U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$
- $U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$
- $U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$
- $U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$
- $U_{10}(x) = 1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 60x^2 - 1$

Δ explicit formulas for the Chebyshev polynomials

• $T_n(x)$

$$= 2^{n-1}\binom{n}{0}x^n - \frac{n}{n-1}2^{n-3}\binom{n-1}{1}x^{n-2}$$

$$+\frac{n}{n-2}2^{n-5}\binom{n-2}{2}x^{n-4} - \frac{n}{n-3}2^{n-7}\binom{n-3}{3}x^{n-6} + \cdots$$

$$= \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor}(-1)^k \frac{n}{n-k}2^{n-2k-1}\binom{n-k}{k}x^{n-2k}$$

wh $n \in$ pos int

• $U_n(x)$

$$= 2^n\binom{n}{0}x^n - 2^{n-2}\binom{n-1}{1}x^{n-2}$$

$$+2^{n-4}\binom{n-2}{2}x^{n-4} - 2^{n-6}\binom{n-3}{3}x^{n-6} + \cdots$$

$$= \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor}(-1)^k 2^{n-2k}\binom{n-k}{k}x^{n-2k}$$

wh $n \in$ nonneg int

Δ note on notation

• T was chosen for CP1 probably because
a transliteration of the Russian name Chebyshev
into German begins with T
eg
Tschebischeff;
there is no ' tee'  in the Russian name
written in the Cyrillic alphabet

• U was chosen for CP2 probably because
U is the alphabetic successor to T

Δ bioline
Pafnuti Lvovich Chebyshev
1821 - 1894
Russian
algebraist, analyst, geometer, number theorist,
probabilist, applied mathematician

$\Delta$ some trig inequalities

let

- $x,\ y \in$ real var
- $x,\ y$ have the geometric interpretation

of being angles measured in radians

then

- $\sin x < x < \tan x \quad \left(0 < x < \frac{\pi}{2}\right)$

- $\tan x < x < \sin x \quad \left(-\frac{\pi}{2} < x < 0\right)$

- $\frac{2}{\pi} < \frac{\sin x}{x} < 1 \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2} \ \&\ x \neq 0\right)$

- $\cos x < \frac{\sin x}{x} < 1 \quad \left(-\pi \leq x \leq \pi \ \&\ x \neq 0\right)$

- $\exists \lim_{x \to 0} \frac{\sin x}{x} \ = 1$

• the Aristarchus inequalities

$$\frac{\sin x}{\sin y} < \frac{x}{y} < \frac{\tan x}{\tan y} \quad \left(0 < y < x < \frac{\pi}{2}\right)$$

Δ bioline
Aristarchus of Samos
ca 310 - ca 230 BCE
Greek
mathematician, astronomer;
first to maintain that
the Earth rotates & revolves around the Sun
& to calculate (by correct geometric argument)
the sizes & distances of the Sun & the Moon,
the results being the best of the time
but inaccurate because more exact numerical observations
were not possible then;
found a more precise value for the length of the solar year;
a lunar crater is named in his honor,
a peak in its center being the brightest formation on the Moon