

A Semi-Systematic Sampling
of Trig Identities: Part III

#25 of Gottschalk's Gestalts

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of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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GG25-2

Δ fcn(A + B) into fcn & cfcn of A & B

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$

- $\cos(A + B) = \cos A \cos B - \sin A \sin B$

- $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\cot A + \cot B}{\cot A \cot B - 1}$

- $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B} = \frac{1 - \tan A \tan B}{\tan A + \tan B}$

- $\sec(A + B) = \frac{\sec A \sec B \csc A \csc B}{\csc A \csc B - \sec A \sec B}$

- $\csc(A + B) = \frac{\sec A \sec B \csc A \csc B}{\sec A \csc B + \csc A \sec B}$

Δ fcn(A - B) into fcn & cfcn of A & B

- $\sin(A - B) = \sin A \cos B - \cos A \sin B$

- $\cos(A - B) = \cos A \cos B + \sin A \sin B$

- $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\cot B - \cot A}{\cot A \cot B + 1}$

- $\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A} = \frac{1 + \tan A \tan B}{\tan A - \tan B}$

- $\sec(A - B) = \frac{\sec A \sec B \csc A \csc B}{\csc A \csc B + \sec A \sec B}$

- $\csc(A - B) = \frac{\sec A \sec B \csc A \csc B}{\sec A \csc B - \csc A \sec B}$

Δ product of sin & cos of 2 angles as sum

- $2 \sin A \sin B = -\cos(A + B) + \cos(A - B)$

- $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

- $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

- $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

Δ product of tan & cot of 2 angles as quotient

$$\begin{aligned} \bullet \tan A \tan B &= \frac{\tan A + \tan B}{\cot A + \cot B} = -\frac{\tan A - \tan B}{\cot A - \cot B} \\ &= \frac{\cos(A - B) - \cos(A + B)}{\cos(A - B) + \cos(A + B)} \end{aligned}$$

$$\begin{aligned} \bullet \tan A \cot B &= \frac{\tan A + \cot B}{\cot A + \tan B} = -\frac{\tan A - \cot B}{\cot A - \tan B} \\ &= \frac{\sin(A + B) + \sin(A - B)}{\sin(A + B) - \sin(A - B)} \end{aligned}$$

$$\begin{aligned} \bullet \cot A \tan B &= \frac{\cot A + \tan B}{\tan A + \cot B} = -\frac{\cot A - \tan B}{\tan A - \cot B} \\ &= \frac{\sin(A + B) - \sin(A - B)}{\sin(A + B) + \sin(A - B)} \end{aligned}$$

$$\begin{aligned} \bullet \cot A \cot B &= \frac{\cot A + \cot B}{\tan A + \tan B} = -\frac{\cot A - \cot B}{\tan A - \tan B} \\ &= \frac{\cos(A - B) + \cos(A + B)}{\cos(A - B) - \cos(A + B)} \end{aligned}$$

Δ sum & difference of sin or cos of 2 angles as product

$$\bullet \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\bullet \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\bullet \cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\bullet \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

Δ sum & difference of tan & cot of 2 angles as quotient

$$\bullet \tan A + \tan B = \frac{\sin(A + B)}{\cos A \cos B}$$

$$\bullet \tan A + \cot B = \frac{\cos(A - B)}{\cos A \sin B}$$

$$\bullet \cot A + \tan B = \frac{\cos(A - B)}{\sin A \cos B}$$

$$\bullet \cot A + \cot B = \frac{\sin(A + B)}{\sin A \sin B}$$

$$\bullet \tan A - \tan B = \frac{\sin(A - B)}{\cos A \cos B}$$

$$\bullet \tan A - \cot B = -\frac{\cos(A + B)}{\cos A \sin B}$$

$$\bullet \cot A - \tan B = \frac{\cos(A + B)}{\sin A \cos B}$$

$$\bullet \cot A - \cot B = -\frac{\sin(A - B)}{\sin A \sin B}$$

Δ sum & difference of sec or csc of 2 angles as product

$$\bullet \sec A + \sec B = 2 \sec A \sec B \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\bullet \sec A - \sec B = 2 \sec A \sec B \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\bullet \csc A + \csc B = 2 \csc A \csc B \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\bullet \csc A - \csc B = -2 \csc A \csc B \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

Δ difference of squares of sin & cos of 2 angles

$$\bullet \sin^2 A - \sin^2 B$$

$$= (\sin A + \sin B)(\sin A - \sin B)$$

$$= \cos^2 B - \cos^2 A$$

$$= (\cos B + \cos A)(\cos B - \cos A)$$

$$= \sin(A + B)\sin(A - B)$$

$$\bullet \sin^2 A - \cos^2 B$$

$$= (\sin A + \cos B)(\sin A - \cos B)$$

$$= \sin^2 B - \cos^2 A$$

$$= (\sin B + \cos A)(\sin B - \cos A)$$

$$= -\cos(A + B)\cos(A - B)$$

$$\begin{aligned} & \bullet \cos^2 A - \sin^2 B \\ &= (\cos A + \sin B)(\cos A - \sin B) \\ &= \cos^2 B - \sin^2 A \\ &= (\cos B + \sin A)(\cos B - \sin A) \\ &= \cos(A + B)\cos(A - B) \end{aligned}$$

$$\begin{aligned} & \bullet \cos^2 A - \cos^2 B \\ &= (\cos A + \cos B)(\cos A - \cos B) \\ &= \sin^2 B - \sin^2 A \\ &= (\sin B + \sin A)(\sin B - \sin A) \\ &= -\sin(A + B)\sin(A - B) \end{aligned}$$

△ quotients of sum & difference of sin & cos of 2 angles

$$\bullet \frac{\sin A - \sin B}{\sin A + \sin B} = \cot \frac{A + B}{2} \tan \frac{A - B}{2}$$

$$\bullet \frac{\cos A + \cos B}{\sin A + \sin B} = \cot \frac{A + B}{2}$$

$$\bullet \frac{\cos A - \cos B}{\sin A + \sin B} = -\tan \frac{A - B}{2}$$

$$\bullet \frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{A + B}{2} \cot \frac{A - B}{2}$$

$$\bullet \frac{\cos A + \cos B}{\sin A - \sin B} = \cot \frac{A - B}{2}$$

$$\bullet \frac{\cos A - \cos B}{\sin A - \sin B} = -\tan \frac{A + B}{2}$$

$$\bullet \frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A + B}{2}$$

$$\bullet \frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A - B}{2}$$

$$\bullet \frac{\cos A - \cos B}{\cos A + \cos B} = -\tan \frac{A + B}{2} \tan \frac{A - B}{2}$$

$$\bullet \frac{\sin A + \sin B}{\cos A - \cos B} = -\cot \frac{A - B}{2}$$

$$\bullet \frac{\sin A - \sin B}{\cos A - \cos B} = -\cot \frac{A + B}{2}$$

$$\bullet \frac{\cos A + \cos B}{\cos A - \cos B} = -\cot \frac{A + B}{2} \cot \frac{A - B}{2}$$

△ quotients of sums & differences of tan & cot of 2 angles

$$\bullet \frac{\tan A + \cot B}{\tan A + \tan B} = \cot B \frac{\cos(A - B)}{\sin(A + B)}$$

$$\bullet \frac{\cot A + \tan B}{\tan A + \tan B} = \cot A \frac{\cos(A - B)}{\sin(A + B)}$$

$$\bullet \frac{\cot A + \cot B}{\tan A + \tan B} = \cot A \cot B$$

$$\bullet \frac{\tan A - \tan B}{\tan A + \tan B} = \frac{\sin(A - B)}{\sin(A + B)}$$

$$\bullet \frac{\tan A - \cot B}{\tan A + \tan B} = -\cot B \cot(A + B)$$

$$\bullet \frac{\cot A - \tan B}{\tan A + \tan B} = \cot A \cot(A + B)$$

$$\bullet \frac{\cot A - \cot B}{\tan A + \tan B} = -\cot A \cot B \frac{\sin(A - B)}{\sin(A + B)}$$

$$\bullet \frac{\tan A + \tan B}{\tan A + \cot B} = \tan B \frac{\sin(A + B)}{\cos(A - B)}$$

$$\bullet \frac{\cot A + \tan B}{\tan A + \cot B} = \cot A \tan B$$

$$\bullet \frac{\cot A + \cot B}{\tan A + \cot B} = \cot A \frac{\sin(A + B)}{\cos(A - B)}$$

$$\bullet \frac{\tan A - \tan B}{\tan A + \cot B} = \tan B \tan(A - B)$$

$$\bullet \frac{\tan A - \cot B}{\tan A + \cot B} = -\frac{\cos(A + B)}{\cos(A - B)}$$

$$\bullet \frac{\cot A - \tan B}{\tan A + \cot B} = \cot A \tan B \frac{\cos(A + B)}{\cos(A - B)}$$

$$\bullet \frac{\cot A - \cot B}{\tan A + \cot B} = -\cot A \tan(A - B)$$

- $\frac{\tan A + \tan B}{\cot A + \tan B} = \tan A \frac{\sin(A + B)}{\cos(A - B)}$

- $\frac{\tan A + \cot B}{\cot A + \tan B} = \tan A \cot B$

- $\frac{\cot A + \cot B}{\cot A + \tan B} = \cot B \frac{\sin(A + B)}{\cos(A - B)}$

- $\frac{\tan A - \tan B}{\cot A + \tan B} = \tan A \tan(A - B)$

- $\frac{\tan A - \cot B}{\cot A + \tan B} = -\tan A \cot B \frac{\cos(A + B)}{\cos(A - B)}$

- $\frac{\cot A - \tan B}{\cot A + \tan B} = \frac{\cos(A + B)}{\cos(A - B)}$

- $\frac{\cot A - \cot B}{\cot A + \tan B} = -\cot B \tan(A - B)$

- $\frac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$
- $\frac{\tan A + \cot B}{\cot A + \cot B} = \tan A \frac{\cos(A - B)}{\sin(A + B)}$
- $\frac{\cot A + \tan B}{\cot A + \cot B} = \tan B \frac{\cos(A - B)}{\sin(A + B)}$
- $\frac{\tan A - \tan B}{\cot A + \cot B} = \tan A \tan B \frac{\sin(A - B)}{\sin(A + B)}$
- $\frac{\tan A - \cot B}{\cot A + \cot B} = -\tan A \cot(A + B)$
- $\frac{\cot A - \tan B}{\cot A + \cot B} = \tan B \cot(A + B)$
- $\frac{\cot A - \cot B}{\cot A + \cot B} = -\frac{\sin(A - B)}{\sin(A + B)}$

- $\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin(A + B)}{\sin(A - B)}$
- $\frac{\tan A + \cot B}{\tan A - \tan B} = \cot B \cot(A - B)$
- $\frac{\cot A + \tan B}{\tan A - \tan B} = \cot A \cot(A - B)$
- $\frac{\cot A + \cot B}{\tan A - \tan B} = \cot A \cot B \frac{\sin(A + B)}{\sin(A - B)}$
- $\frac{\tan A - \cot B}{\tan A - \tan B} = -\cot B \frac{\cos(A + B)}{\sin(A - B)}$
- $\frac{\cot A - \tan B}{\tan A - \tan B} = \cot A \frac{\cos(A + B)}{\sin(A - B)}$
- $\frac{\cot A - \cot B}{\tan A - \tan B} = -\cot A \cot B$

- $\frac{\tan A + \tan B}{\tan A - \cot B} = -\tan B \tan(A + B)$

- $\frac{\tan A + \cot B}{\tan A - \cot B} = -\frac{\cos(A - B)}{\cos(A + B)}$

- $\frac{\cot A + \tan B}{\tan A - \cot B} = -\cot A \tan B \frac{\cos(A - B)}{\cos(A + B)}$

- $\frac{\cot A + \cot B}{\tan A - \cot B} = -\cot A \tan(A + B)$

- $\frac{\tan A - \tan B}{\tan A - \cot B} = -\tan B \frac{\sin(A - B)}{\cos(A + B)}$

- $\frac{\cot A - \tan B}{\tan A - \cot B} = -\cot A \tan B$

- $\frac{\cot A - \cot B}{\tan A - \cot B} = \cot A \frac{\sin(A - B)}{\cos(A + B)}$

- $\frac{\tan A + \tan B}{\cot A - \tan B} = \tan A \tan (A + B)$
- $\frac{\tan A + \cot B}{\cot A - \tan B} = \tan A \cot B \frac{\cos(A - B)}{\cos(A + B)}$
- $\frac{\cot A + \tan B}{\cot A - \tan B} = \frac{\cos(A - B)}{\cos(A + B)}$
- $\frac{\cot A + \cot B}{\cot A - \tan B} = \cot B \tan (A + B)$
- $\frac{\tan A - \tan B}{\cot A - \tan B} = \tan A \frac{\sin(A - B)}{\cos(A + B)}$
- $\frac{\tan A - \cot B}{\cot A - \tan B} = -\tan A \cot B$
- $\frac{\cot A - \cot B}{\cot A - \tan B} = -\cot B \frac{\sin(A - B)}{\cos(A + B)}$

$$\bullet \frac{\tan A + \tan B}{\cot A - \cot B} = -\tan A \tan B \frac{\sin(A + B)}{\sin(A - B)}$$

$$\bullet \frac{\tan A + \cot B}{\cot A - \cot B} = -\tan A \cot(A - B)$$

$$\bullet \frac{\cot A + \tan B}{\cot A - \cot B} = -\tan B \cot(A - B)$$

$$\bullet \frac{\cot A + \cot B}{\cot A - \cot B} = -\frac{\sin(A + B)}{\sin(A - B)}$$

$$\bullet \frac{\tan A - \tan B}{\cot A - \cot B} = -\tan A \tan B$$

$$\bullet \frac{\tan A - \cot B}{\cot A - \cot B} = \tan A \frac{\cos(A + B)}{\sin(A - B)}$$

$$\bullet \frac{\cot A - \tan B}{\cot A - \cot B} = -\tan B \frac{\cos(A + B)}{\sin(A - B)}$$

△ quotient of sum & difference of sec & csc of 2 angles

$$\bullet \frac{\sec A - \sec B}{\sec A + \sec B} = \tan \frac{A + B}{2} \tan \frac{A - B}{2}$$

$$\bullet \frac{\csc A + \csc B}{\sec A + \sec B} = \cot A \cot B \tan \frac{A + B}{2}$$

$$\bullet \frac{\csc A - \csc B}{\sec A + \sec B} = -\cot A \cot B \tan \frac{A - B}{2}$$

- $\frac{\sec A + \sec B}{\sec A - \sec B} = \cot \frac{A + B}{2} \cot \frac{A - B}{2}$
- $\frac{\csc A + \csc B}{\sec A - \sec B} = \cot A \cot B \cot \frac{A - B}{2}$
- $\frac{\csc A - \csc B}{\sec A - \sec B} = -\cot A \cot B \cot \frac{A + B}{2}$

$$\bullet \frac{\sec A + \sec B}{\csc A + \csc B} = \tan A \tan B \cot \frac{A + B}{2}$$

$$\bullet \frac{\sec A - \sec B}{\csc A + \csc B} = \tan A \tan B \tan \frac{A - B}{2}$$

$$\bullet \frac{\csc A - \csc B}{\csc A + \csc B} = -\cot \frac{A + B}{2} \tan \frac{A - B}{2}$$

$$\bullet \frac{\sec A + \sec B}{\csc A - \csc B} = -\tan A \tan B \cot \frac{A - B}{2}$$

$$\bullet \frac{\sec A - \sec B}{\csc A - \csc B} = -\tan A \tan B \tan \frac{A + B}{2}$$

$$\bullet \frac{\csc A + \csc B}{\csc A - \csc B} = -\tan \frac{A + B}{2} \cot \frac{A - B}{2}$$

△ some two - angle identities

- $\sin A + \sin B + \sin(A + B)$

$$= 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{A + B}{2}$$

- $\cos A + \cos B + \cos(A + B)$

$$= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{A + B}{2} - 1$$

- $\frac{\sin(A - B) + \sin A + \sin(A + B)}{\cos(A - B) + \cos A + \cos(A + B)} = \frac{\sin A}{\cos A} = \tan A$

- $\frac{\cos(A - B) + \cos A + \cos(A + B)}{\sin(A - B) + \sin A + \sin(A + B)} = \frac{\cos A}{\sin A} = \cot A$

(note that $A - B$, A , $A + B$ are in arithmetic progression;
iow A in the arithmetic mean of $A + B$ and $A - B$)

$$\bullet \sin^2(A + B)$$

$$= \sin^2 A + \sin^2 B + 2 \sin A \sin B \cos(A + B)$$

$$\bullet \cos^2(A + B)$$

$$= \cos^2 A + \cos^2 B - 2 \sin A \sin B \cos(A + B) - 1$$