A Semi-Systematic Sampling of Trig Identities: Part II

#24 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms of the Organization & Exposition of Mathematics
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Δ sums & differences of powers of sinA & cosA

- $\sin A + \cos A = \dfrac{\tan A - \cot A}{\sec A - \csc A} = \dfrac{\sec A + \csc A}{\tan A + \cot A}$

- $\sin A - \cos A = \dfrac{\tan A - \cot A}{\sec A + \csc A} = \dfrac{\sec A - \csc A}{\tan A + \cot A}$

- $\sin^2 A + \cos^2 A = 1$

- $\sin^2 A - \cos^2 A = -\cos 2A$

- $\sin^3 A \pm \cos^3 A = (\sin A \pm \cos A)\left(1 \mp \dfrac{\sin 2A}{2}\right)$

- $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A = -\cos 2A$

Δ some higher - power - function formulas for sin & cos

- $\sin A = \sin A$

- $\sin^2 A = \frac{1}{2}(-\cos 2A + 1)$

- $\sin^3 A = \frac{1}{4}(-\sin 3A + 3\sin A)$

- $\sin^4 A = \frac{1}{8}(\cos 4A - 4\cos 2A + 3)$

- $\sin^5 A = \frac{1}{16}(\sin 5A - 5\sin 3A + 10\sin A)$

- $\sin^6 A = \frac{1}{32}(-\cos 6A + 6\cos 4A - 15\cos 2A + 10)$

- $\sin^7 A = \frac{1}{64}(-\sin 7A + 7\sin 5A - 21\sin 3A + 35\sin A)$

- $\cos A = \cos A$

- $\cos^2 A = \frac{1}{2}(\cos 2A + 1)$

- $\cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$

- $\cos^4 A = \frac{1}{8}(\cos 4A + 4\cos 2A + 3)$

- $\cos^5 A = \frac{1}{16}(\cos 5A + 5\cos 3A + 10\cos A)$

- $\cos^6 A = \frac{1}{32}(\cos 6A + 6\cos 4A + 15\cos 2A + 10)$

- $\cos^7 A = \frac{1}{64}(\cos 7A + 7\cos 5A + 21\cos 3A + 35\cos A)$

Δ the general higher - power formulas
for sin & cos
($n \in$ pos int)

- $$\sin^{2n} A = \frac{1}{2^{2n}}\left[\sum_{k=0}^{n-1}(-1)^{n-k}\, 2\binom{2n}{k}\cos 2(n-k)A + \binom{2n}{n}\right]$$

- $$\sin^{2n-1} A = \frac{1}{2^{2n-2}}\sum_{k=0}^{n-1}(-1)^{n+k-1}\binom{2n-1}{k}\sin(2n-2k-1)A$$

- $$\cos^{2n} A = \frac{1}{2^{2n}}\left[\sum_{k=0}^{n-1} 2\binom{2n}{k}\cos 2(n-k)A + \binom{2n}{n}\right]$$

- $$\cos^{2n-1} A = \frac{1}{2^{2n-2}}\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos(2n-2k-1)A$$

Δ a quartet of similarly defined trig functions
viz

$$\sec A + \tan A$$

$$\sec A - \tan A$$

$$\csc A + \cot A$$

$$\csc A - \cot A$$

which consists of
two pairs of reciprocal trig functions;
the two pairs begin analogously
in their alternative expressions
but then diverge

- $\sec A + \tan A = \dfrac{1}{\sec A - \tan A}$

$$= \frac{1 + \sin A}{\cos A} = \frac{\cos A}{1 - \sin A}$$

$$= \frac{\csc A + 1}{\cot A} = \frac{\cot A}{\csc A - 1}$$

$$= \sqrt{\frac{1 + \sin A}{1 - \sin A}} = \sqrt{\frac{\csc A + 1}{\csc A - 1}}$$

$$= \sqrt{\frac{\cot A + \cos A}{\cot A - \cos A}} = \sqrt{\frac{\sec A + \tan A}{\sec A - \tan A}}$$

$$= \frac{\cos\frac{A}{2} + \sin\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}} = \frac{\csc\frac{A}{2} + \sec\frac{A}{2}}{\csc\frac{A}{2} - \sec\frac{A}{2}}$$

$$= \frac{1 + \tan\frac{A}{2}}{1 - \tan\frac{A}{2}} = \frac{\cot\frac{A}{2} + 1}{\cot\frac{A}{2} - 1}$$

$$= \tan\left(\frac{A}{2} + \frac{\pi}{4}\right)$$

$$= \exp \mathrm{gd}^{-1} A$$

= exponential of inverse gudermannian of A

- $\sec A - \tan A = \dfrac{1}{\sec A + \tan A}$

$$= \frac{1 - \sin A}{\cos A} = \frac{\cos A}{1 + \sin A}$$

$$= \frac{\csc A - 1}{\cot A} = \frac{\cot A}{\csc A + 1}$$

$$= \sqrt{\frac{1 - \sin A}{1 + \sin A}} = \sqrt{\frac{\csc A - 1}{\csc A + 1}}$$

$$= \sqrt{\frac{\cot A - \cos A}{\cot A + \cos A}} = \sqrt{\frac{\sec A - \tan A}{\sec A + \tan A}}$$

$$= \frac{\cos\frac{A}{2} - \sin\frac{A}{2}}{\cos\frac{A}{2} + \sin\frac{A}{2}} = \frac{\csc\frac{A}{2} - \sec\frac{A}{2}}{\csc\frac{A}{2} + \sec\frac{A}{2}}$$

$$= \frac{1 - \tan\frac{A}{2}}{1 + \tan\frac{A}{2}} = \frac{\cot\frac{A}{2} - 1}{\cot\frac{A}{2} + 1}$$

$$= \cot\left(\frac{A}{2} + \frac{\pi}{4}\right)$$

$$= \exp\left(-\mathrm{gd}^{-1} A\right)$$

= exponential of minus inverse gudermannian of A

- $\csc A + \cot A = \dfrac{1}{\csc A - \cot A}$

$$= \frac{1+\cos A}{\sin A} = \frac{\sin A}{1-\cos A}$$

$$= \frac{\sec A + 1}{\tan A} = \frac{\tan A}{\sec A - 1}$$

$$= \sqrt{\frac{1+\cos A}{1-\cos A}} = \sqrt{\frac{\sec A + 1}{\sec A - 1}}$$

$$= \sqrt{\frac{\tan A + \sin A}{\tan A - \sin A}} = \sqrt{\frac{\csc A + \cot A}{\csc A - \cot A}}$$

$$= \cot\frac{A}{2}$$

- $\csc A - \cot A = \dfrac{1}{\csc A + \cot A}$

$$= \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A}$$

$$= \frac{\sec A - 1}{\tan A} = \frac{\tan A}{\sec A + 1}$$

$$= \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \sqrt{\frac{\sec A - 1}{\sec A + 1}}$$

$$= \sqrt{\frac{\tan A - \sin A}{\tan A + \sin A}} = \sqrt{\frac{\csc A - \cot A}{\csc A + \cot A}}$$

$$= \tan \frac{A}{2}$$

Δ fcnA $\pm$ cfcnA

- $\sin A + \cos A = \sqrt{2}\sin\left(A + \frac{\pi}{4}\right) = \sqrt{2}\cos\left(A - \frac{\pi}{4}\right)$

- $\sin A - \cos A = \sqrt{2}\sin\left(A - \frac{\pi}{4}\right) = -\sqrt{2}\cos\left(A + \frac{\pi}{4}\right)$

- $\tan A + \cot A = \sec A \csc A = 2\csc 2A$

- $\tan A - \cot A = -2\cot 2A$

- $\sec A + \csc A = 2(\sin A + \cos A)\csc 2A$

- $\sec A - \csc A = 2(\sin A - \cos A)\csc 2A$

Δ fcnA & cfcnA & squares

• $\sin^2 A + \cos^2 A = 1$

• $\sin^2 A - \cos^2 A$

$= (\sin A + \cos A)(\sin A - \cos A)$

$= -\cos 2A$

• $(\sin A \pm \cos A)^2 = 1 \pm \sin 2A$

• $\tan^2 A + \cot^2 A$

$= \sec^2 A + \csc^2 A - 2$

$= \sec^2 A \csc^2 A - 2$

$= 4\csc^2 2A - 2$

• $\tan^2 A - \cot^2 A$

$= (\tan A + \cot A)(\tan A - \cot A)$

$= \sec^2 A - \csc^2 A$

$= (\sec A + \csc A)(\sec A - \csc A)$

$= -4\cot 2A \csc 2A$

- $(\tan A + \cot A)^2$

$= \sec^2 A + \csc^2 A$

$= \sec^2 A \csc^2 A$

$= 4\csc^2 2A$

- $(\tan A - \cot A)^2$

$= \sec^2 A + \csc^2 A - 4$

$= \sec^2 A \csc^2 A - 4$

$= 4\csc^2 2A - 4$

$= 4\cot^2 2A$

- $(\sec A \pm \csc A)^2 = 4\csc 2A\,(\csc 2A \pm 1)$

Δ fcn$\left(A + n\pi\right)$ $(n \in \text{int})$

- $\sin\left(A + n\pi\right) = (-1)^n \sin A$
- $\cos\left(A + n\pi\right) = (-1)^n \cos A$
- $\tan\left(A + n\pi\right) = \tan A$
- $\cot\left(A + n\pi\right) = \cot A$
- $\sec\left(A + n\pi\right) = (-1)^n \sec A$
- $\csc\left(A + n\pi\right) = (-1)^n \csc A$

$$\Delta \ \text{fcn}\left(A + [2n+1]\frac{\pi}{2} \right) \ (n \in \text{int})$$

- $\sin\left(A + [2n+1]\frac{\pi}{2} \right) = (-1)^n \cos A$

- $\cos\left(A + [2n+1]\frac{\pi}{2} \right) = (-1)^{n+1} \sin A$

- $\tan\left(A + [2n+1]\frac{\pi}{2} \right) = -\cot A$

- $\cot\left(A + [2n+1]\frac{\pi}{2} \right) = -\tan A$

- $\sec\left(A + [2n+1]\frac{\pi}{2} \right) = (-1)^{n+1} \csc A$

- $\csc\left(A + [2n+1]\frac{\pi}{2} \right) = (-1)^n \sec A$

$\Delta \ \text{fcn}\left(A + \frac{\pi}{4}\right)$

- $\sin\left(A + \frac{\pi}{4}\right) = \frac{\cos A + \sin A}{\sqrt{2}}$
- $\cos\left(A + \frac{\pi}{4}\right) = \frac{\cos A - \sin A}{\sqrt{2}}$
- $\sec\left(A + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{\cos A - \sin A}$
- $\csc\left(A + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{\cos A + \sin A}$

- $\tan\left(A + \frac{\pi}{4}\right)$

$$= \frac{1 + \tan A}{1 - \tan A} = \frac{\cot A + 1}{\cot A - 1}$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A} = \frac{\csc A + \sec A}{\csc A - \sec A}$$

$$= \frac{1 + \sin 2A}{\cos 2A} = \frac{\cos 2A}{1 - \sin 2A}$$

$$= \frac{\csc 2A + 1}{\cot 2A} = \frac{\cot 2A}{\csc 2A - 1}$$

$$= \sec 2A + \tan 2A = \frac{1}{\sec 2A - \tan 2A}$$

$$= \sqrt{\frac{1+\sin 2A}{1-\sin 2A}} = \sqrt{\frac{\csc 2A+1}{\csc 2A-1}}$$

$$= \sqrt{\frac{\cot 2A+\cos 2A}{\cot 2A-\cos 2A}} = \sqrt{\frac{\sec 2A+\tan 2A}{\sec 2A-\tan 2A}}$$

$$= \exp \text{gd}^{-1} 2A$$

= exponential of inverse gudermannian of 2A

- $\cot\left(A + \frac{\pi}{4}\right)$

$$= \frac{1 - \tan A}{1 + \tan A} = \frac{\cot A - 1}{\cot A + 1}$$

$$= \frac{\cos A - \sin A}{\cos A + \sin A} = \frac{\csc A - \sec A}{\csc A + \sec A}$$

$$= \frac{1 - \sin 2A}{\cos 2A} = \frac{\cos 2A}{1 + \sin 2A}$$

$$= \frac{\csc 2A - 1}{\cot 2A} = \frac{\cot 2A}{\csc 2A + 1}$$

$$= \sec 2A - \tan 2A = \frac{1}{\sec 2A + \tan 2A}$$

$$= \sqrt{\frac{1 - \sin 2A}{1 + \sin 2A}} = \sqrt{\frac{\csc 2A - 1}{\csc 2A + 1}}$$

$$= \sqrt{\frac{\cot 2A - \cos 2A}{\cot 2A + \cos 2A}} = \sqrt{\frac{\sec 2A - \tan 2A}{\sec 2A + \tan 2A}}$$

$$= \exp\left(-\operatorname{gd}^{-1} 2\,A\right)$$

= exponential of minus inverse gudermannian of 2A

Δ a real linear combination of sinA & cosA expressed as a single function

- $a \sin A + b \cos A$

$$= \sqrt{a^2 + b^2} \sin\left(A + \tan^{-1} \frac{b}{a} \right)$$

$$= \sqrt{a^2 + b^2} \cos\left(A - \cot^{-1} \frac{b}{a} \right)$$

Δ some related single - angle identities

$$\bullet \quad \frac{1+\sin A}{\cos A} + \frac{\cos A}{1+\sin A}$$

$$= \frac{1-\sin A}{\cos A} + \frac{\cos A}{1-\sin A}$$

$$= \frac{\csc A+1}{\cot A} + \frac{\cot A}{\csc A+1}$$

$$= \frac{\csc A-1}{\cot A} + \frac{\cot A}{\csc A-1}$$

$$= 2\sec A$$

- $\dfrac{1+\sin A}{\cos A} - \dfrac{\cos A}{1+\sin A}$

$$= \frac{\cos A}{1-\sin A} - \frac{1-\sin A}{\cos A}$$

$$= \frac{\csc A+1}{\cot A} - \frac{\cot A}{\csc A+1}$$

$$= \frac{\cot A}{\csc A-1} - \frac{\csc A-1}{\cot A}$$

$$= 2\tan A$$

- $$\frac{1+\cos A}{\sin A} + \frac{\sin A}{1+\cos A}$$

$$= \frac{1-\cos A}{\sin A} + \frac{\sin A}{1-\cos A}$$

$$= \frac{\sec A+1}{\tan A} + \frac{\tan A}{\sec A+1}$$

$$= \frac{\sec A-1}{\tan A} + \frac{\tan A}{\sec A-1}$$

$$= 2\csc A$$

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- $\dfrac{1+\cos A}{\sin A} - \dfrac{\sin A}{1+\cos A}$

$= \dfrac{\sin A}{1-\cos A} - \dfrac{1-\cos A}{\sin A}$

$= \dfrac{\sec A+1}{\tan A} - \dfrac{\tan A}{\sec A+1}$

$= \dfrac{\tan A}{\sec A-1} - \dfrac{\sec A-1}{\tan A}$

$= 2\cot A$

Δ just for fun: a few 6 - packs

- $\sin^2 A + \cos^2 A + \tan^2 A + \cot^2 A + \sec^2 A + \csc^2 A$

$= \; 8\cot^2 2A + 7$

- $\sin^2 A - \cos^2 A + \tan^2 A - \cot^2 A + \sec^2 A - \csc^2 A$

$= \; -\cos 2A\,(8\csc^2 2A + 1)$

- $\sin A - \cos A + \tan A - \cot A + \sec A - \csc A$

$= \; \tan A \sec A - \cot A \csc A - 2\cot 2A$

- $\sin A \cos A \tan A \cot A \sec A \csc A \; = \; 1$

- $\sin A \div \cos A \div \tan A \div \cot A \div \sec A \div \csc A$

$= \; \sin^3 A \sec A \; = \; \sin^2 A \tan A$

- $$\frac{\sin A}{\cos A} + \frac{\tan A}{\cot A} + \frac{\sec A}{\csc A} = \tan^2 A + 2\tan A$$

- $$\frac{\sin A}{\cos A}\,\frac{\tan A}{\cot A}\,\frac{\sec A}{\csc A} = \tan^4 A$$

Δ sin A times fcn A

- $\sin^2 A = \dfrac{1}{\csc^2 A}$
- $\sin A \cos A = \dfrac{1}{2}\sin 2A$
- $\sin A \tan A = \sec A - \cos A$
- $\sin A \cot A = \cos A$
- $\sin A \sec A = \tan A$
- $\sin A \csc A = 1$

Δ cos A times fcn A

- $\cos A \sin A = \frac{1}{2} \sin 2A$
- $\cos^2 A = \frac{1}{\sec^2 A}$
- $\cos A \tan A = \sin A$
- $\cos A \cot A = \csc A - \sin A$
- $\cos A \sec A = 1$
- $\cos A \csc A = \cot A$

Δ tan A times fcn A

- $\tan A \sin A = \sec A - \cos A$
- $\tan A \cos A = \sin A$
- $\tan^2 A = \dfrac{1}{\cot^2 A}$
- $\tan A \cot A = 1$
- $\tan A \sec A = \dfrac{1}{\csc A - \sin A}$
- $\tan A \csc A = \sec A$

Δ cot A times fcn A

- $\cot A \sin A = \cos A$
- $\cot A \cos A = \csc A - \sin A$
- $\cot A \tan A = 1$
- $\cot^2 A = \dfrac{1}{\tan^2 A}$
- $\cot A \sec A = \csc A$
- $\cot A \csc A = \dfrac{1}{\sec A - \cos A}$

Δ sec A times fcn A

- $\sec A \sin A = \tan A$
- $\sec A \cos A = 1$
- $\sec A \tan A = \dfrac{1}{\csc A - \sin A}$
- $\sec A \cot A = \csc A$
- $\sec^2 A = \dfrac{1}{\cos^2 A}$
- $\sec A \csc A = 2 \csc 2A$

Δ csc A times fcn A

- $\csc A \sin A = 1$
- $\csc A \cos A = \cot A$
- $\csc A \tan A = \sec A$
- $\csc A \cot A = \dfrac{1}{\sec A - \cos A}$
- $\csc A \sec A = 2 \csc 2A$
- $\csc^2 A = \dfrac{1}{\sin^2 A}$

Δ a sequence of related single - angle identities

- $\tan A + 2\cot 2A = \cot A$

- $\tan A + 2\tan 2A + 4\cot 4A = \cot A$

- $\tan A + 2\tan 2A + 4\tan 4A + 8\cot 8A = \cot A$

- $\tan A + 2\tan 2A + 4\tan 4A + 8\tan 8A + 16\cot 16A = \cot A$

- etc

note: this sequence follows from
the repeated application of the first identity

Δ exact square-root representation of trig fcn values

T. let
- n $\in$ integer
- A $\in$ angle whose measure is n degrees

then
tfsape

(1) all defined six basic trig fcns of A
are expressible ito
finitely many applications
of the operations of
addition, subtraction, multiplication, division
& extraction of square roots,
starting with the integers
& staying in the real field

(2) some one of the defined six basic trig fcns of A
is so expressible

(3) n is an integer multiple of 3

eg

- $\sin 3^{o} = \cos 87^{o}$

$= \dfrac{1}{16}\left(-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{15+3\sqrt{5}}\right)$

$= 0.05233\ 59562\cdots$

- $\cos 3^{o} = \sin 87^{o}$

$= \dfrac{1}{16}\left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{15+3\sqrt{5}}\right)$

$= 0.99862\ 95348\cdots$

- $\sin 18^{o} = \cos 72^{o}$

$= \dfrac{1}{4}\left(-1+\sqrt{5}\right)$

$= 0.30901\ 69944\cdots$

- $\cos 18^{o} = \sin 72^{o}$

$= \dfrac{1}{4}\sqrt{10+2\sqrt{5}}$

$= 0.95105\ 65163\cdots$