Trig Patterns

#21 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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GG21-1 (39)

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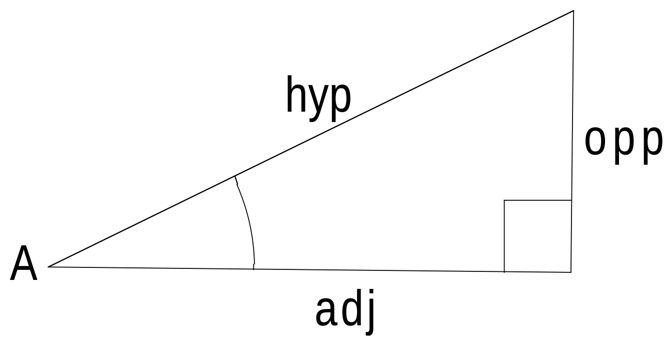
□ trig patterns
in diagrams
&
in words:
they help
in understanding
&
in remembering

Δ the definitions of the six trig fcns of an acute angle of a right triangle

hyp = hypotenuse
adj = adjacent side
opp = opposite side



- $\sin A = \dfrac{\text{opp}}{\text{hyp}}$
- $\cos A = \dfrac{\text{adj}}{\text{hyp}}$
- $\tan A = \dfrac{\text{opp}}{\text{adj}}$
- $\cot A = \dfrac{\text{adj}}{\text{opp}}$
- $\sec A = \dfrac{\text{hyp}}{\text{adj}}$
- $\csc A = \dfrac{\text{hyp}}{\text{opp}}$

Δ this trig mnemonic

is a pronounceable manufactured ‘word’
for the definitions
of the sine, cosine, tangent
of an acute angle
of a right triangle

The Great Trig Chief

• SOHCAHTOA

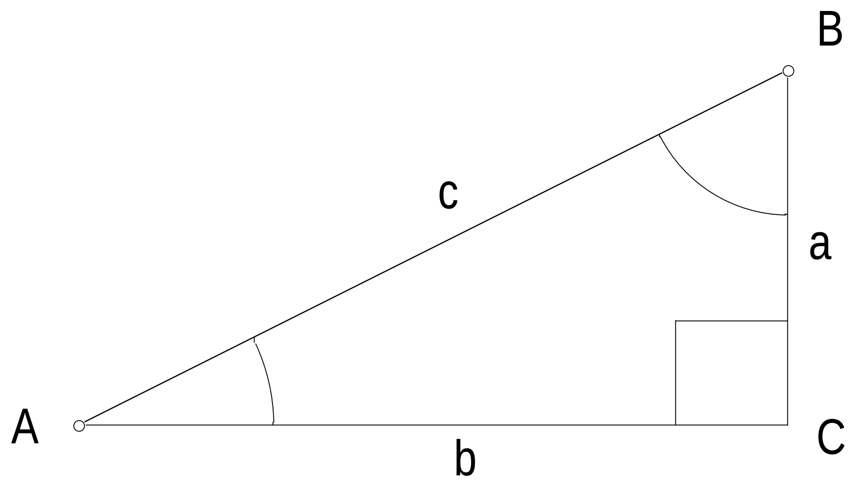
says

sin = opp/hyp

cos = adj/hyp

tan = opp/adj

Δ fcn of angle = cofcn of complement

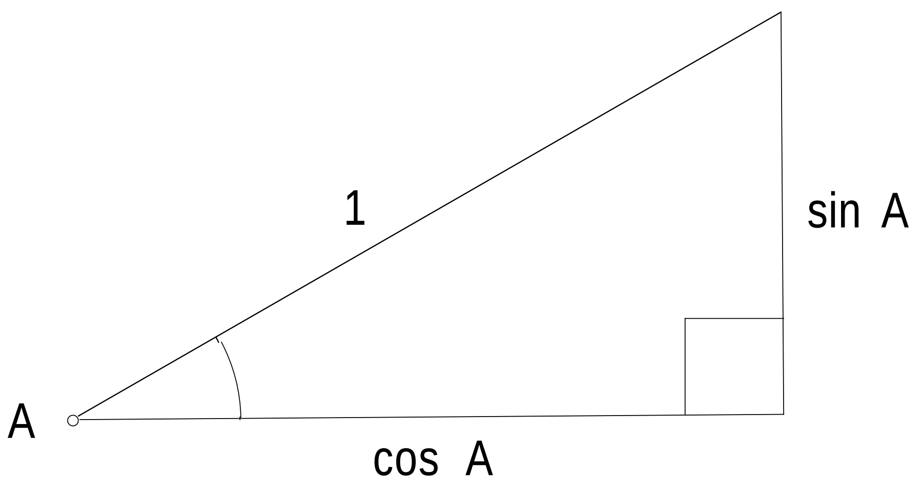


- $A + B = 90^{o} = \frac{\pi^{r}}{2}$

- $\sin A = \frac{a}{c} = \cos B$
- $\cos A = \frac{b}{c} = \sin B$
- $\tan A = \frac{a}{b} = \cot B$
- $\cot A = \frac{b}{a} = \tan B$
- $\sec A = \frac{c}{b} = \csc B$
- $\csc A = \frac{c}{a} = \sec B$

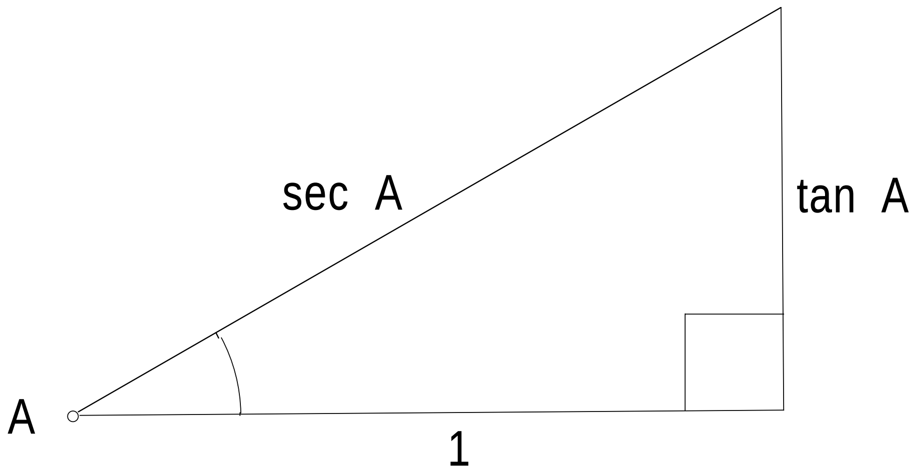
Δ the three pythagorean identities
are easily deducible from
the pythagorean theorem
as follows:

1st PID



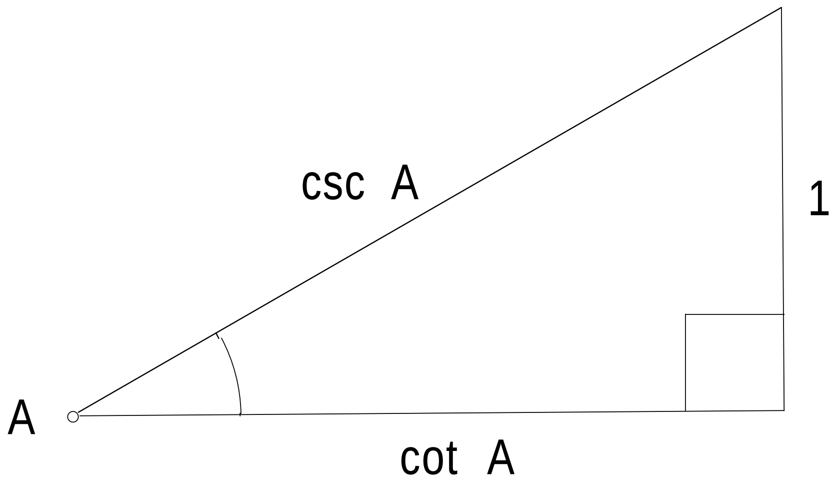
- $\sin^2 A + \cos^2 A = 1$

2nd PID



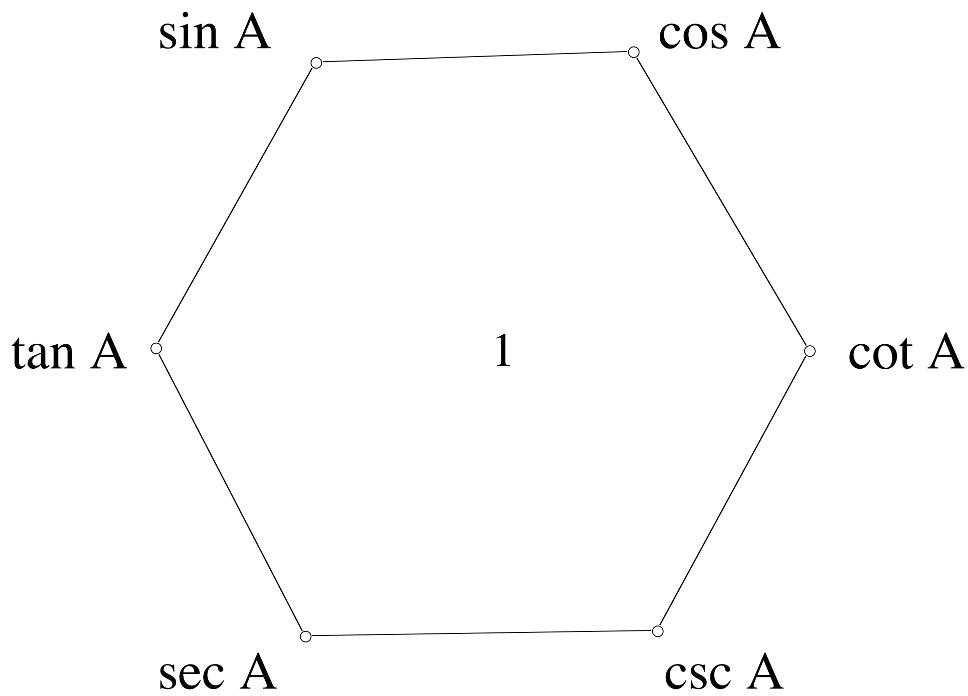
- $1 + \tan^2 A = \sec^2 A$

3rd PID



- $1 + \cot^2 A = \csc^2 A$

∆ the trig hex



• level corners are cofunctions;
unco- fcns on left; co- fcns on right

• opposite corners are reciprocal functions

• any corner
= the product of the two adjacent corners

• any corner
= the quotient of either adjacent corner
divided by
the remote corner in the same direction

• the product of any two nonadjacent corners
is the middle function (using 1 for opposite corners)

• alternate pairs of adjacent corners
beginning with the top pair
are connected by a pythagorean identity;
think of three pointing-down triangles
with vertices at
sin A, cos A, 1
&
tan A , 1, sec A
&
1, cot A, csc A;
in each triangle
the sum of the squares of the top vertices
equals the square of the bottom vertex

Δ the six trig fcns of the three special angles

$$30^o = \frac{\pi^r}{6}$$

$$45^o = \frac{\pi^r}{4}$$

$$60^o = \frac{\pi^r}{3}$$

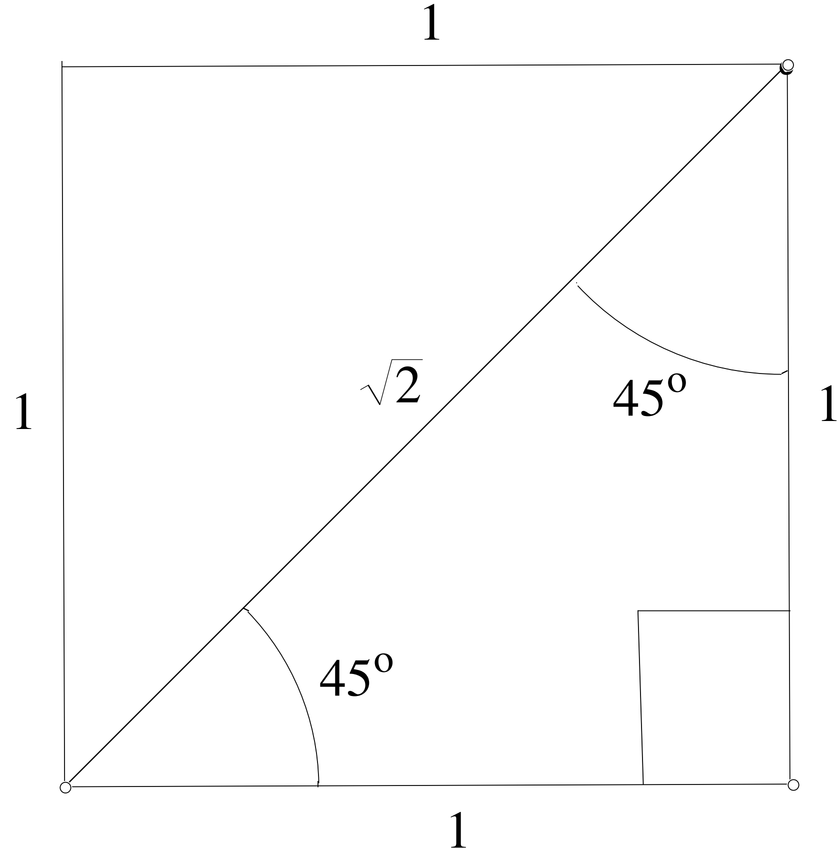
are readily computed by

starting with
a square whose sides = 1
&
an equilateral triangle whose sides = 2

and then
splitting them in half
viz
splitting the square
with a diagonal
&
splitting the triangle
with an altitude

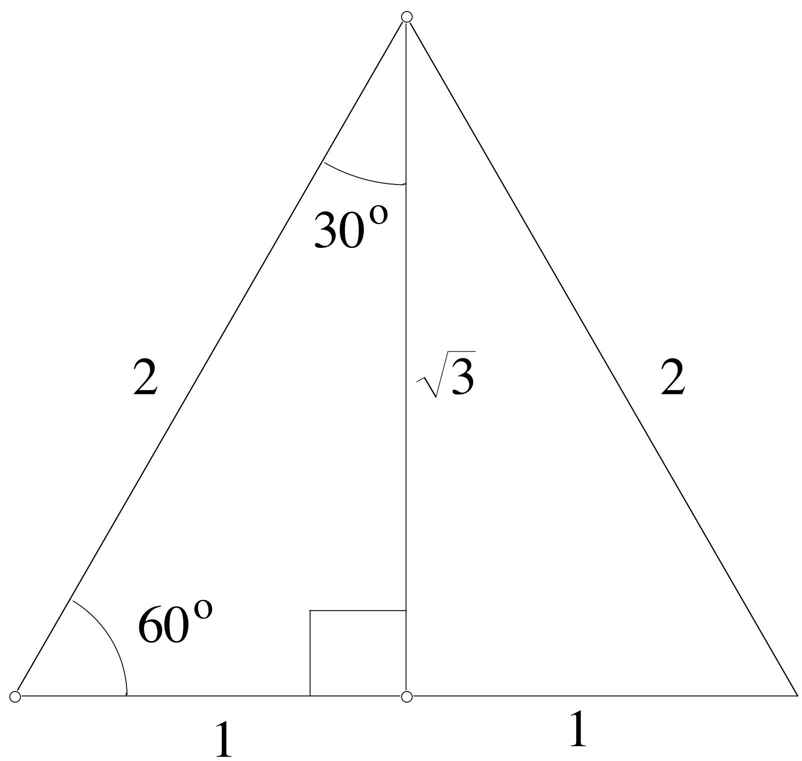
• in more detail:

to find the trig fcn values of 45^{o},
draw a diagonal
of a square with sides of length 1
&
look at one of the triangles formed
viz

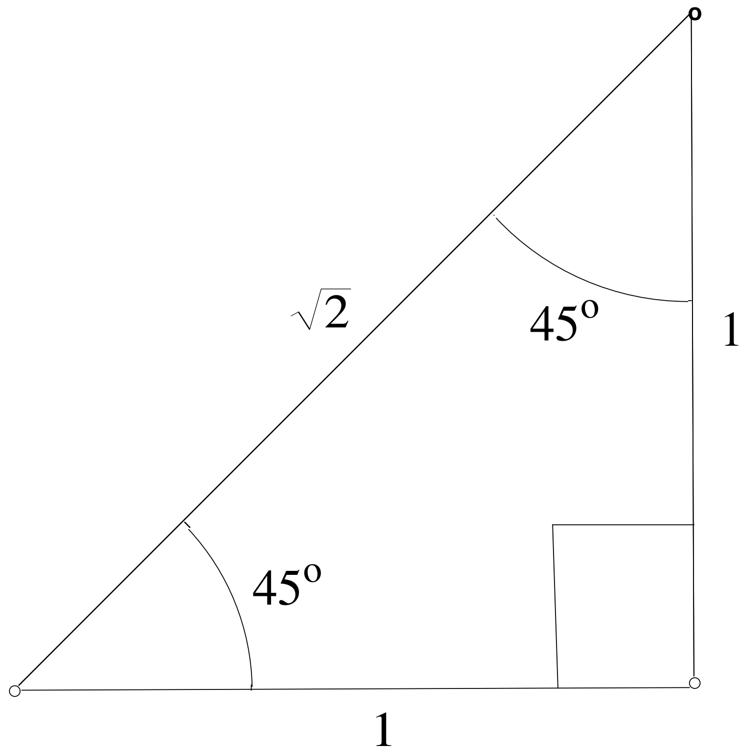


• in more detail:

to find the trig fcn values of 30^o & 60^o,
draw an altitude
of an equilateral triangle with sides of length 2
&
look at one of the triangles formed
viz



• to find the trig fcn values of 45°,
visualize the $45^\circ - 45^\circ - 90^\circ$ triangle
with opposite sides $1,\ 1,\ \sqrt{2}$



and read off the fcns

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

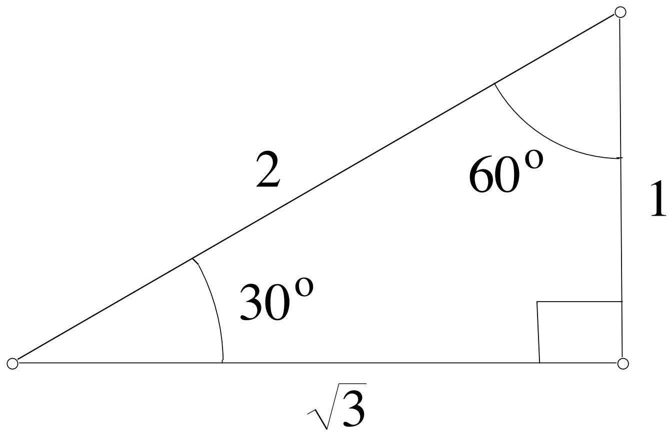
$$\tan 45^\circ = 1$$

$$\cot 45^\circ = 1$$

$$\sec 45^\circ = \sqrt{2}$$

$$\csc 45^\circ = \sqrt{2}$$

• to find the trig fcn values of 30^o & 60^o,
visualize the $30^o - 60^o - 90^o$ triangle
with opposite sides $1,\ \sqrt{3},\ 2$



and read off the fcns

$$\sin 30^o = \frac{1}{2} = \cos 60^o$$

$$\cos 30^o = \frac{\sqrt{3}}{2} = \sin 60^o$$

$$\tan 30^o = \frac{1}{\sqrt{3}} = \cot 60^o$$

$$\cot 30^o = \sqrt{3} = \tan 60^o$$

$$\sec 30^o = \frac{2}{\sqrt{3}} = \csc 60^o$$

$$\csc 30^o = 2 = \sec 60^o$$

Δ pretty patterns of squares of values
of trig fcns of special angles

A^o	0	30	45	60	90
$\sin^2 A$	$\frac{0}{4}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$\cos^2 A$	$\frac{4}{4}$	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{0}{4}$
$\tan^2 A$	$\frac{0}{4}$	$\frac{1}{3}$	$\frac{2}{2}$	$\frac{3}{1}$	$\frac{4}{0}$
$\cot^2 A$	$\frac{4}{0}$	$\frac{3}{1}$	$\frac{2}{2}$	$\frac{1}{3}$	$\frac{0}{4}$
$\sec^2 A$	$\frac{4}{4}$	$\frac{4}{3}$	$\frac{4}{2}$	$\frac{4}{1}$	$\frac{4}{0}$
$\csc^2 A$	$\frac{4}{0}$	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$

many trig identities are recognizable
in the preceding table

eg

• rows reverse for cofcns
viz
sin & cos
tan & cot
sec & csc

• fractions invert for reciprocal functions
viz
sin & csc
tan & cot
sec & csc

• the three pythagorean identities appear

• for secant square & cosecant square,
their sum equals their product

note: the four ‘fractions’ in which
zero occurs in the denominator
are to be viewed formally
ie as expressions without meaning
since
division by zero is undefined and sensibly undefinable

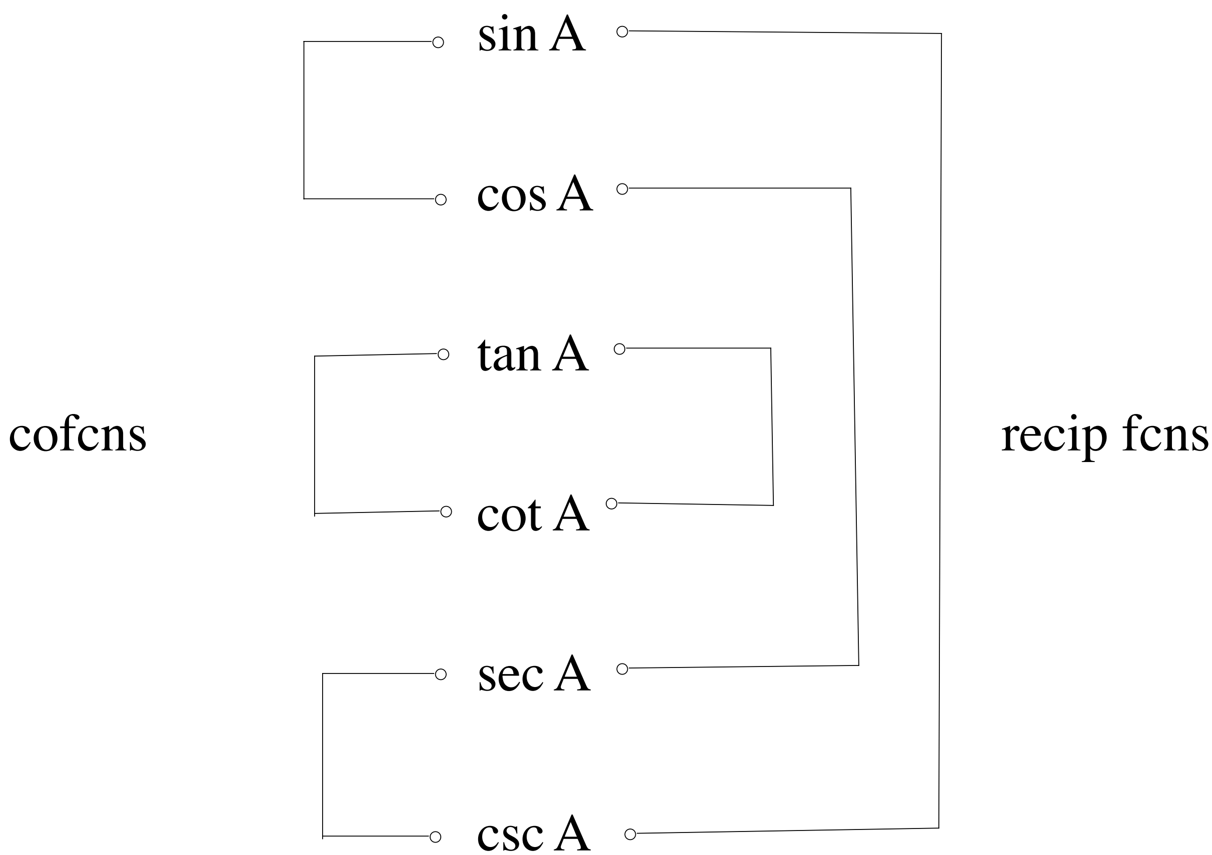
if division by zero
is defined in any manner,
then the usual laws of operations with numbers
would fail to be valid;
consequently,
any laws governing the arithmetic operations
would have to exclude divison by zero;
the net effect is that division by zero is ‘defined’
but then can never be used which means that
a sensible definition of division by zero does not exist;
here are a few examples to illustrate this argument

$\frac{a\,x}{a} = x$ holds for $a \neq 0$ but fails when $a = 0$

$a\,x = a\,y \;\Rightarrow\; x = y$ holds for $a \neq 0$ but fails when $a = 0$

$\frac{a}{b} = c \;\Leftrightarrow\; a = b\,c$ holds for $b \neq 0$ but fails when $b = 0$

Δ cofunctions & reciprocal functions



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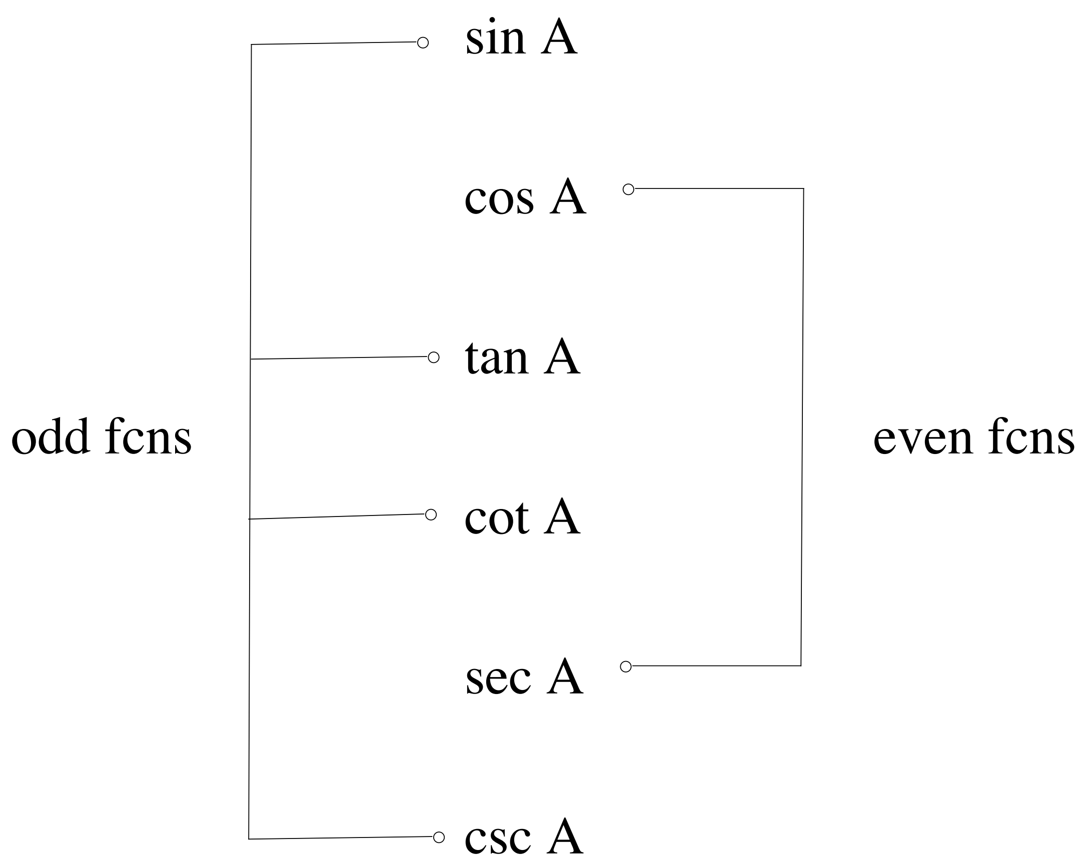
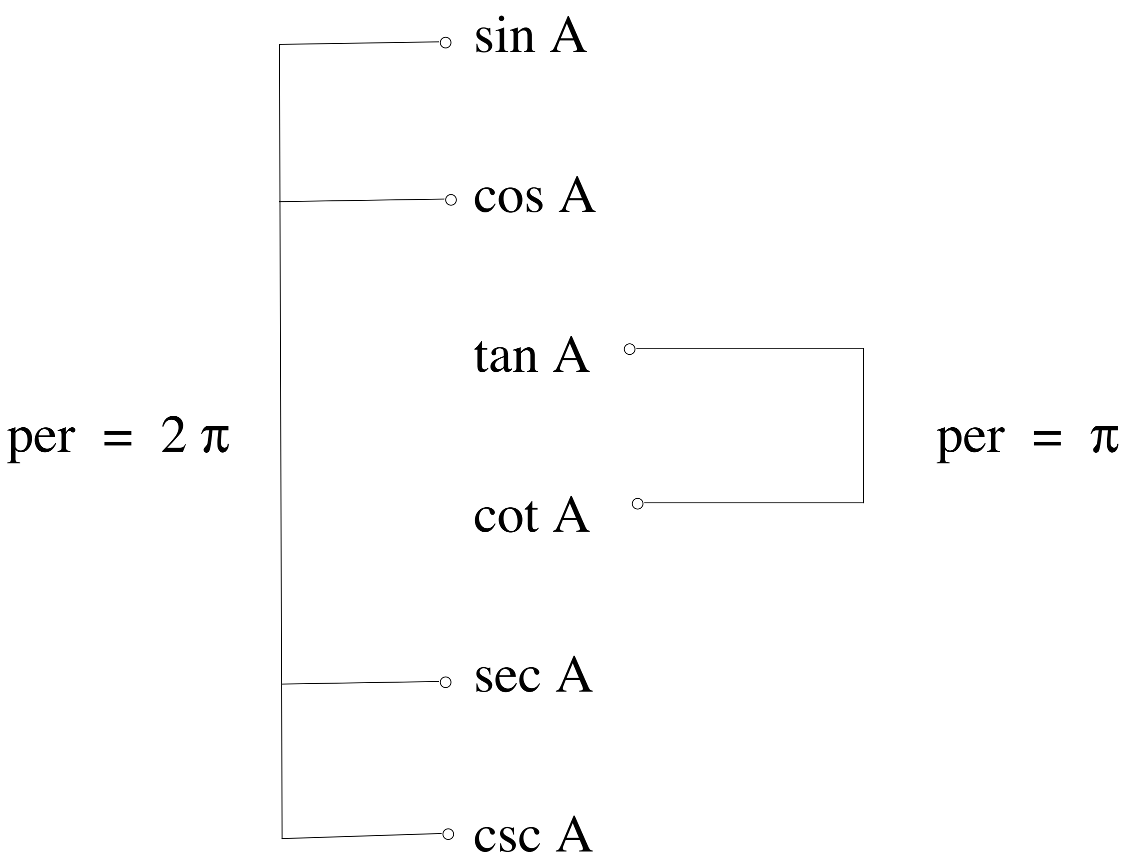
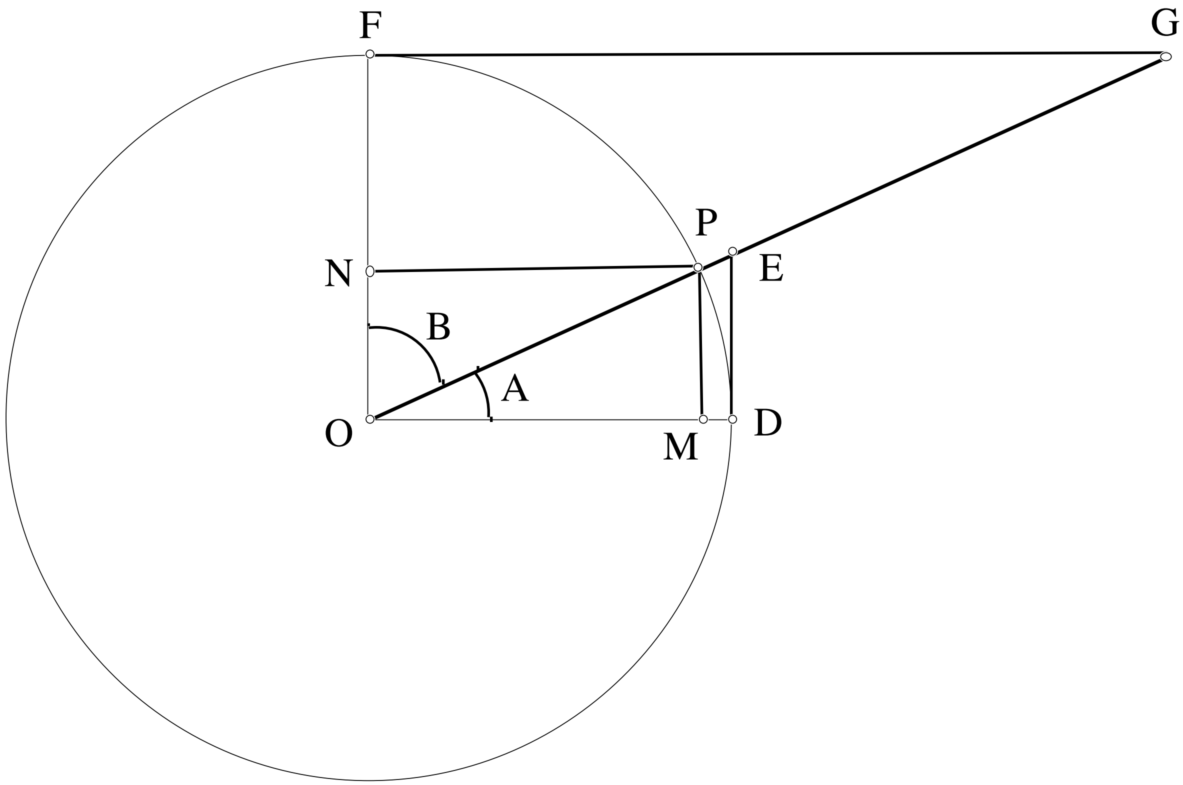
Δ parity of trig fcns



Δ periodicity of trig fcns



Δ segment lengths
associated with a unit circle
as trig fcns



$\overline{OP} = \overline{OD} = \overline{OF} = 1$

$OD \perp OF$

$PM \perp OD$ & $PN \perp OF$

$DE \perp OD$ & $FG \perp OF$

$\overline{MP} = \overline{ON} = \sin A = \cos B$

$\overline{NP} = \overline{OM} = \cos A = \sin B$

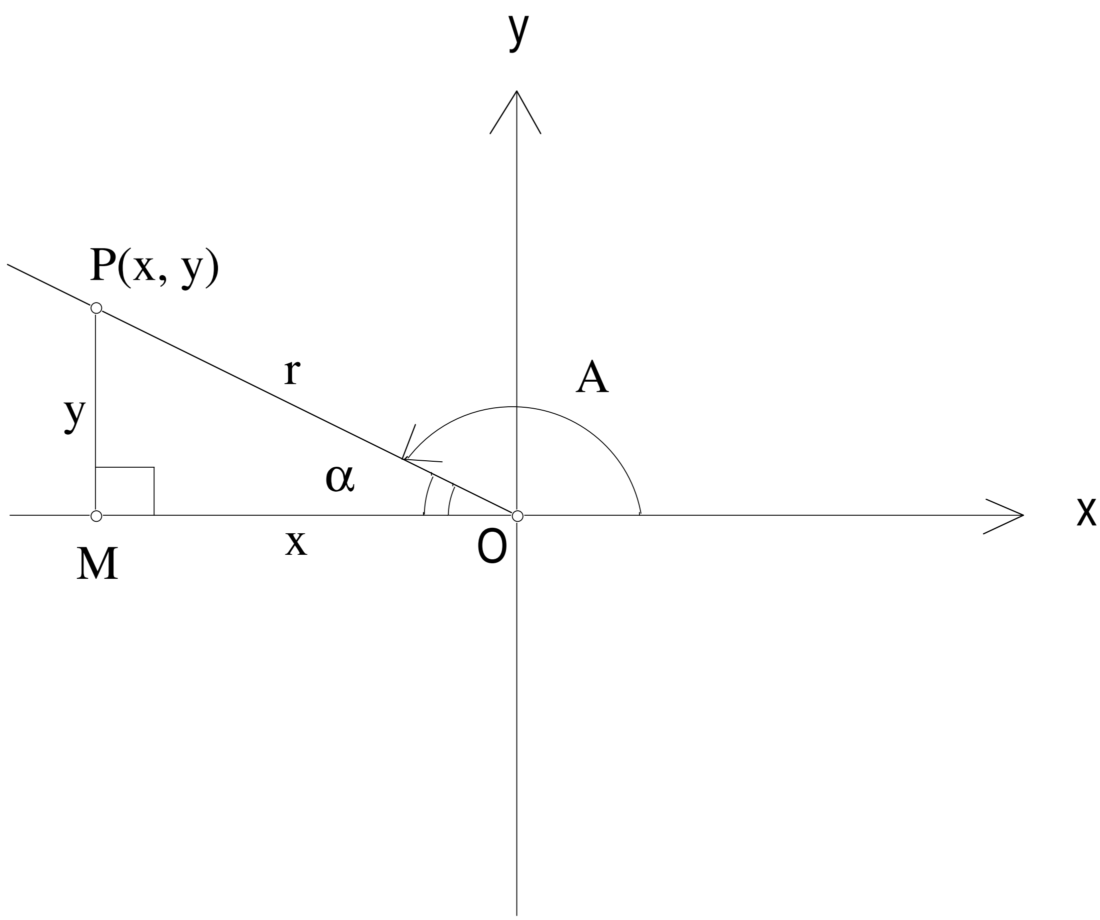
$\overline{DE} = \tan A = \cot B$

$\overline{FG} = \cot A = \tan B$

$\overline{OE} = \sec A = \csc B$

$\overline{OG} = \csc A = \sec B$

Δ to be looked at
& then
visualized



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∆ it is of the utmost importance in understanding math
to be able to see with the mind's eye,
to visualize;
here is an example;
just think as you read
and do not write anything down

• think of coordinate axes,
the x-axis horizontal and pointing to the right,
the y-axis vertical and pointing upward,
the axes intersecting at the origin O

• a rotary angle A
of any size and any sign
is placed in standard position
with the vertex at the origin O,
with the initial side along the positive x-axis,
and
with the terminal side falling somewhere in some quadrant

• choose any point P on the terminal side

• let the coordinates of P be
x = abs for abscissa and y = ord for ordinate
and think P(x, y)

• let r = rad be the radial distance of P
ie
r = rad is the distance $\overline{OP}$ from the origin O to the point P

• we are now in a position to define
the six basic trig functions of the rotary angle A
viz

$$\sin A = \frac{\text{ord}}{\text{rad}} = \frac{y}{r}$$

$$\cos A = \frac{\text{abs}}{\text{rad}} = \frac{x}{r}$$

$$\tan A = \frac{\text{ord}}{\text{abs}} = \frac{y}{x}$$

$$\cot A = \frac{\text{abs}}{\text{ord}} = \frac{x}{y}$$

$$\sec A = \frac{\text{rad}}{\text{abs}} = \frac{r}{x}$$

$$\csc A = \frac{\text{rad}}{\text{ord}} = \frac{r}{y}$$

• to associate
a right triangle
and
an acute angle
with the rotary angle A,
draw a perpendicular
from the point P on the terminal side
to the x-axis at the point M;
there is formed a right triangle OMP
with right angle at M;
the right triangle OMP
is called the residual right triangle
and its acute angle POM
with vertex at the origin O
is called the residual angle α
of the rotary angle A;
the six basic trig functions of A
are equal numerically to
the same six basic trig functions of α;

in the residual right triangle OMP

the horizontal leg $\overline{OM} = |x|$

the vertical leg $\overline{MP} = |y|$

the hypotenuse $\overline{OP} = r$

and

by the pythagorean theorem

$$r^2 = x^2 + y^2$$

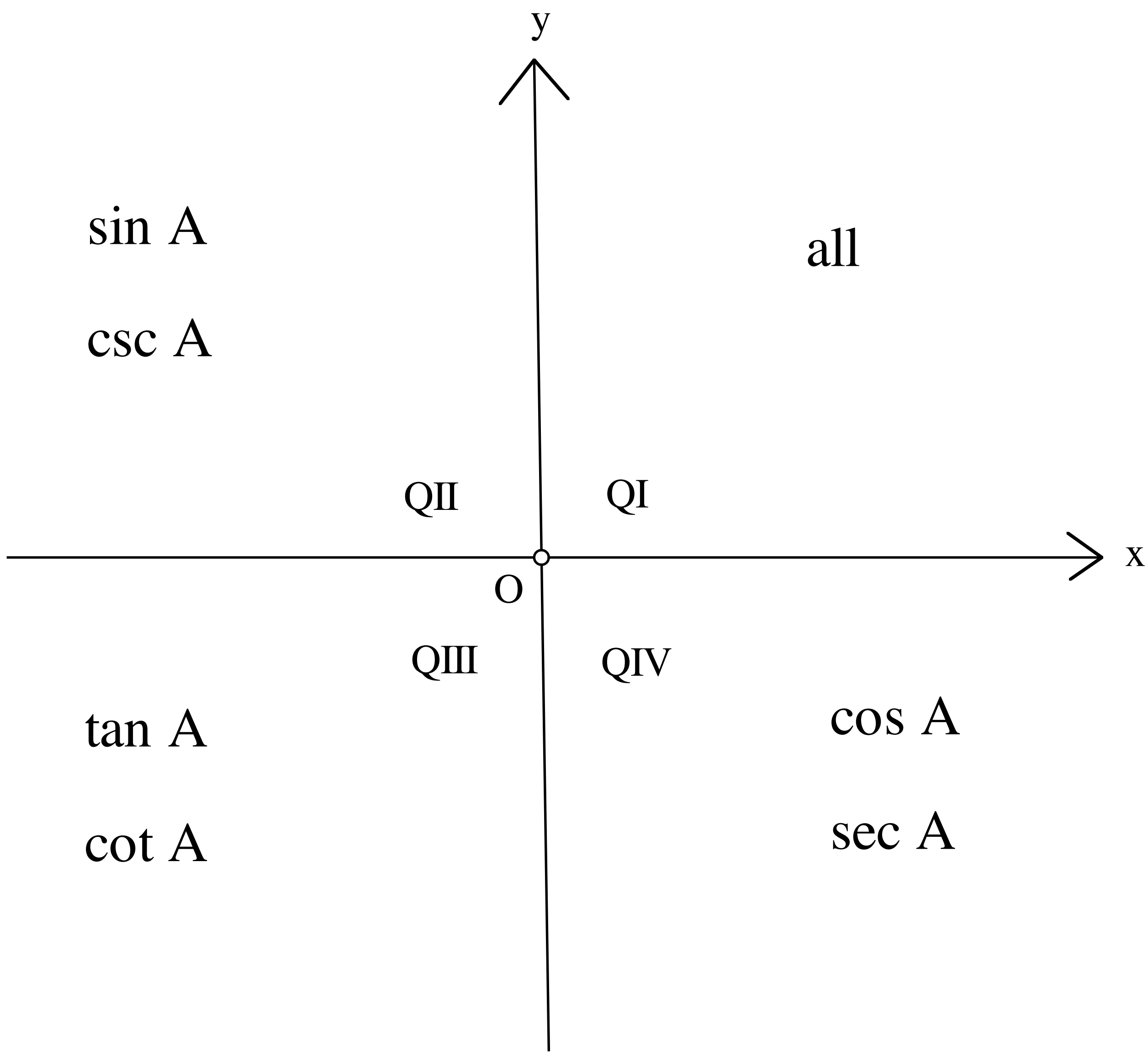
&

$$r = \sqrt{x^2 + y^2}$$

note: quadrantal angles offer
simpler diagrams & equations

Δ signs of trig fcns by quadrant

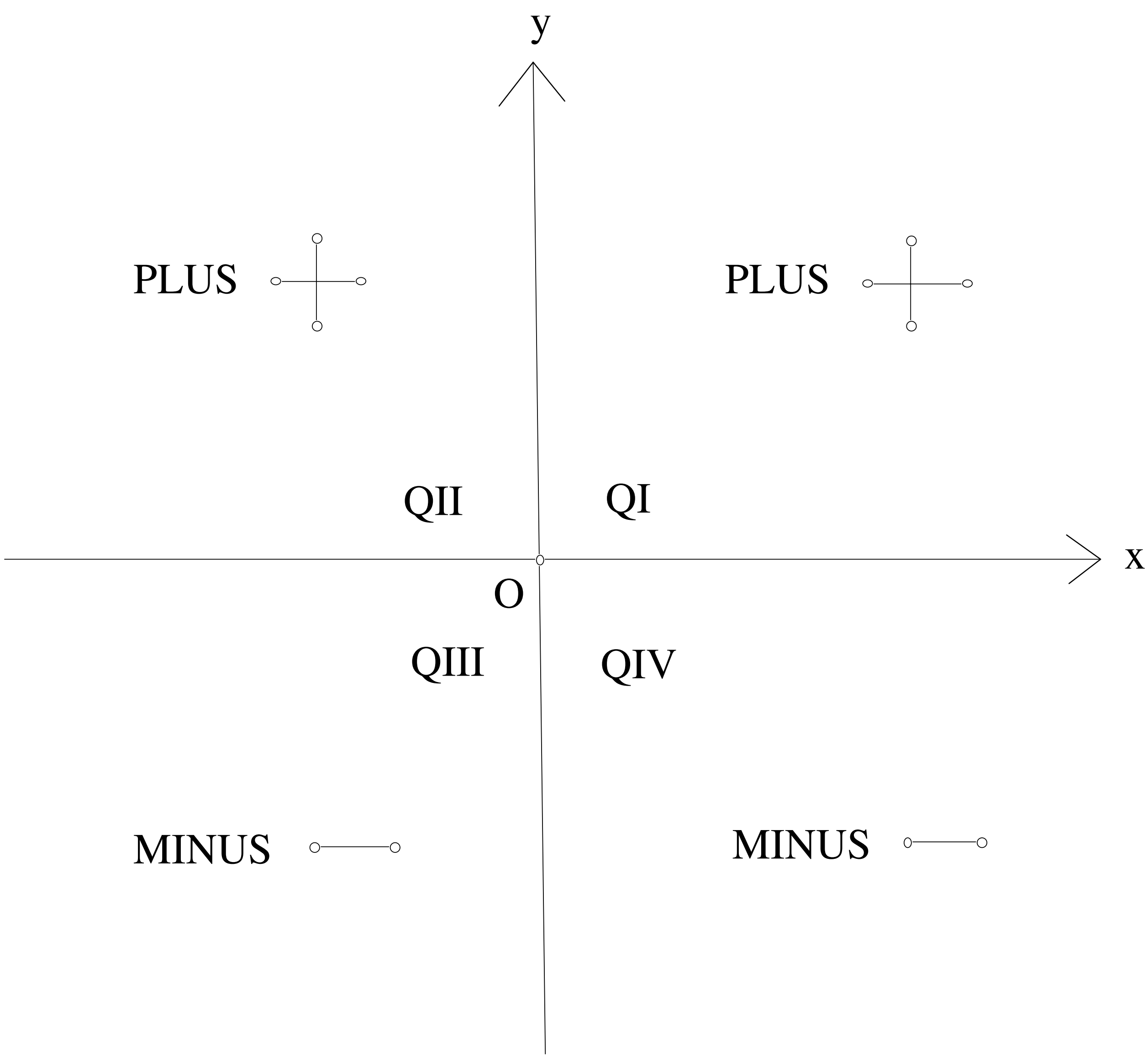
• positivity of trig fcns by quadrant



as a mnemonic
note that the initial letters of
cos, all, sin, tan
spell the word
CAST

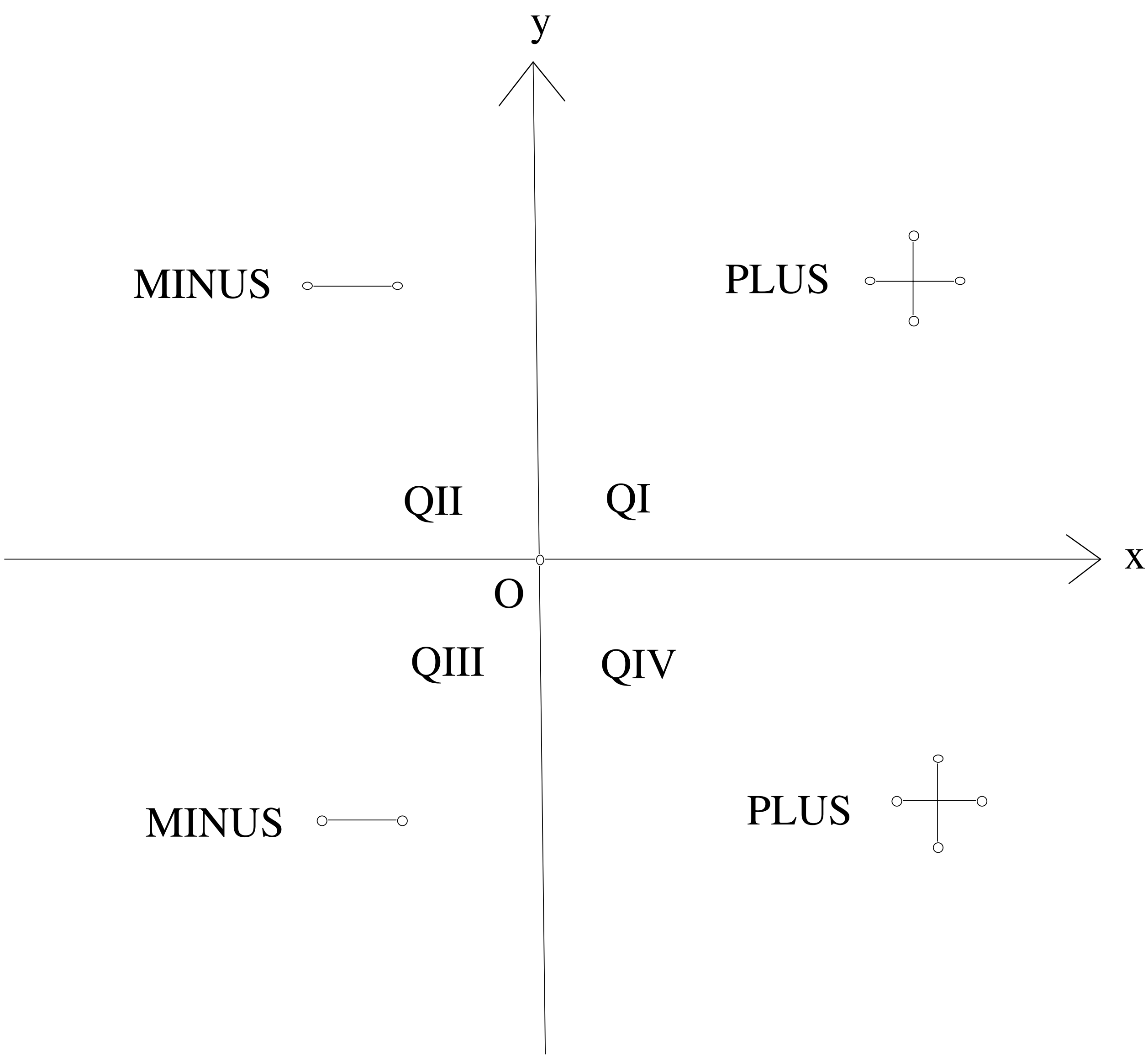
• signs of

$$\sin A = \frac{y}{r} \ \& \ \csc A = \frac{r}{y}$$



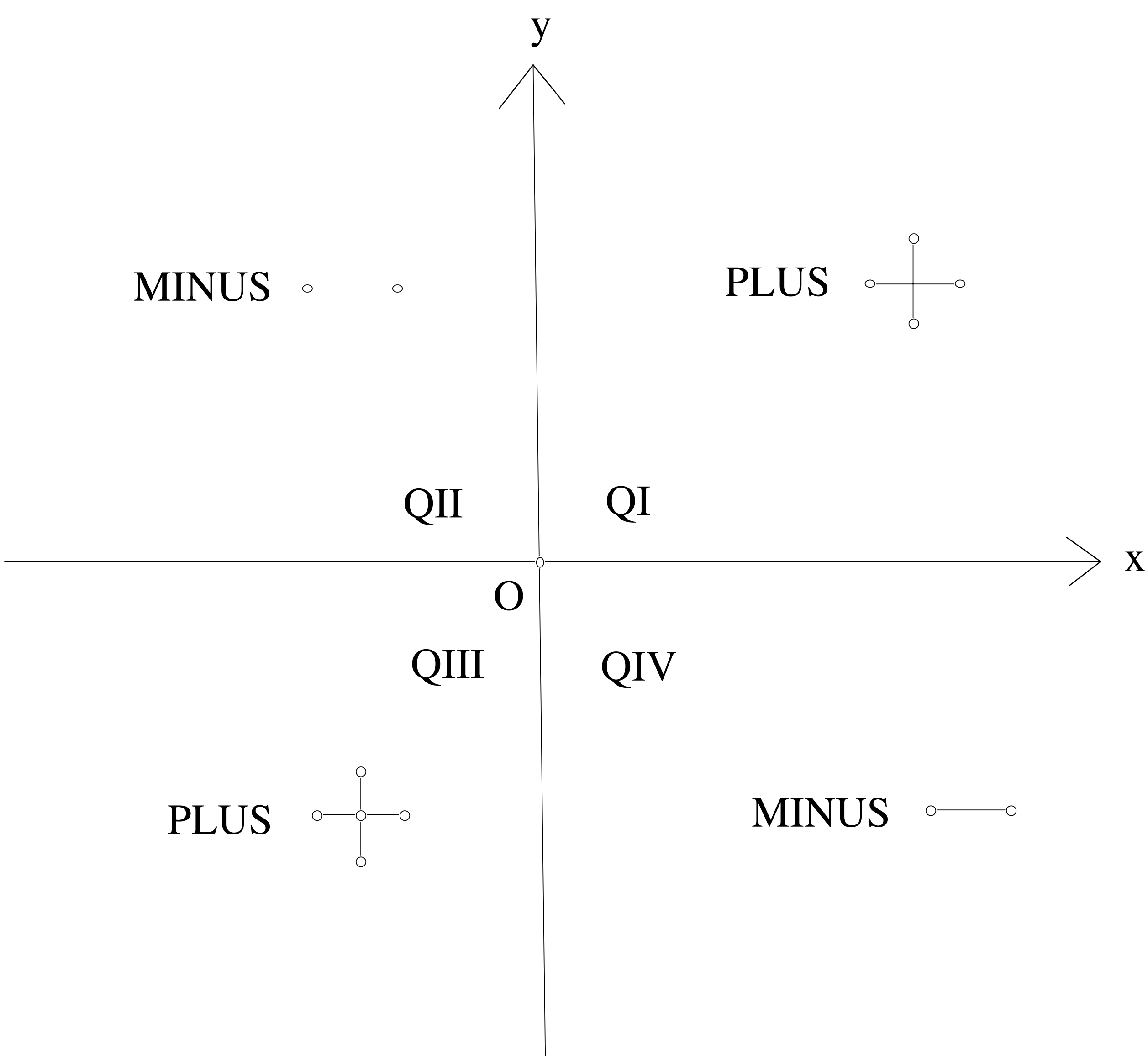
• signs of

$$\cos A = \frac{x}{r} \ \& \ \sec A = \frac{r}{x}$$



• signs of

$$\tan A = \frac{x}{y} \ \& \ \cot A = \frac{y}{x}$$

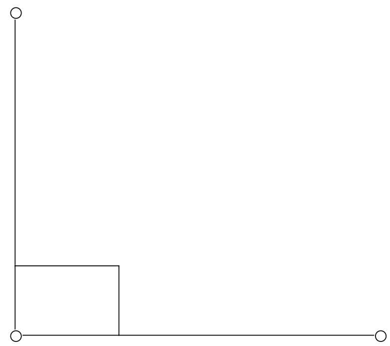


Δ on quadrantal angles

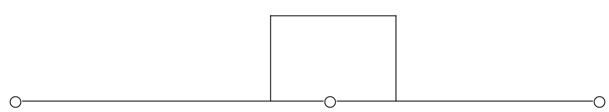
a quadrantal rotary / sectorial angle
$=_{df}$ a rotary / sectorial angle
whose measure is an integral multiple
of $90^o = \dfrac{\pi^r}{2}$

the four nonzero quadrantal sectorial angles
in name, abbreviation, measure, and diagram
are

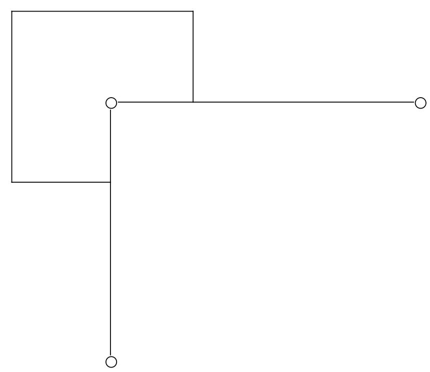
• right angle = rt $\angle$ = 90^{o} = $\frac{\pi^{r}}{2}$



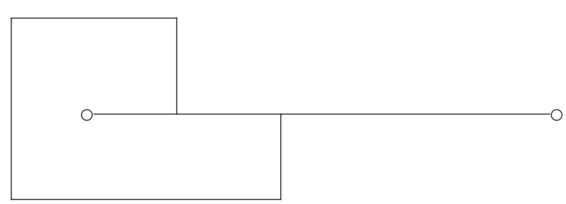
• straight angle = str $\angle$ = 180^{o} = π^{r}



• reflex - right angle = rrt ∠ = 270^o = $\frac{3\pi^r}{2}$



• round angle = rnd ∠ = 360^o = $2\pi^r$



Δ the interesting etymology of the word ‘sine’

• the etymology of ‘sine’ traces
a line of development of trigonometry:
from India in the Sanskrit language
to Middle Age Arabic-speaking Arabia,
from Arabian Baghdad to Arabian Spain,
meeting scholarly Latin in Spain,
transmitted in Latin to western Europe,
and entering English
and the other modern European languages

• the English word ‘sine’
comes from the Latin word ‘sinus’;
the Latin word ‘sinus’ = ‘sine’ was first used
in its modern trig sense ca 1150 by

Gerard of Cremona
1114 - 1187
Italian, worked in Spain
translator from Arabic to Latin
of many scholarly manuscripts

• in India ca 510

Aryabhata
ca 476 - ca 550
Indian
mathematician, astronomer

used the Sanskrit word ‘jya-ardha’ meaning ‘chord-half’
(our sine essentially)
and then abbreviated it to
jya = jiva = chord = bowstring;
this word with its math meaning was adopted into Arabic as
the meaningless word ‘jiba’
which has the same sound as
the Sanskrit word ‘jya = jiva’;
the word ‘jiba’ was written as ‘jb’
since written Arabic omits vowels
and relies only on consonants;
later Arabic writers used ‘jaib’ also written as ‘jb’
in place of ‘jiba’;
now ‘jaib’ is meaningful and means
bay, bosom, breast,
the hanging fold of a toga about the breast,
the bosom of a garment;
the Latin word ‘sinus’ means many things such as
bay, bending, bosom, breast, curve, gulf, hollow
and it also means ‘garment-fold’ as does the Arabic ‘jaib’;
thus ‘jaib’ and ‘sinus’ have much in common;
Gerard chose ‘sinus’ as the translation of ‘jb’,
and our word ‘sine’ came to be