Quaternions & Octonions Made Easy

#20 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2001

GG20-1 (22)

500 Angell St #414
Providence RI 02906

GG20-2

□ quaternions

• here is
a glimpse of things to come,
a quick peak into the future,
a ten-minute course on quaternions
based on 3-dimensional vector analysis

• a scalar = df a real number

• a vector = df a real 3-vector

• a quaternion = df an ordered pair of a scalar & a vector

• it is easy to define
the linear structure
= the vector space structure of quaternions $\mathbb{H}$
viz
a 4-dimensional vector space over the real field
with the four basis elements

$$(1, \mathbf{0}) = (1, 0, 0, 0)$$
$$(0, \mathbf{i}) = (0, 1, 0, 0)$$
$$(0, \mathbf{j}) = (0, 0, 1, 0)$$
$$(0, \mathbf{k}) = (0, 0, 0, 1)$$

where
all linear operations are defined componentwise

• it is harder to define
multiplication for quaternions
which will turn the quaterions into
a division ring
& indeed
a real associative linear algebra
with noncommutative multiplication

• it is more suggestive & notationally simpler
to think of a quaternion as a ' formal sum'
viz
quaternion
$= q$
= scalar plus vector
$= \alpha + \mathbf{a}$
where α is a scalar & $\mathbf{a}$ is a vector
instead of
the precise ordered pair definition
quaternion
$= q$
= ordered pair of scalar & vector
$= (\alpha, \mathbf{a})$
$= (\alpha, \mathbf{0}) + (0, \mathbf{a})$

• we will want
the multiplication of quaternions to be bilinear
& therefore to define the product of any two quaternions
it is enuf to define the product of any two of
the four canonical basis quaternions
1, **i**, **j**, **k**

• take the multiplication table for
1, **i**, **j**, **k**
to be by definition

1 is the multiplicative bilateral identity element

$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$

$\mathbf{i}\mathbf{j} = \mathbf{k}$ & $\mathbf{j}\mathbf{i} = -\mathbf{k}$

$\mathbf{j}\mathbf{k} = \mathbf{i}$ & $\mathbf{k}\mathbf{j} = -\mathbf{i}$

$\mathbf{k}\mathbf{i} = \mathbf{j}$ & $\mathbf{i}\mathbf{k} = -\mathbf{j}$

• the quaternion product of two vectors
is given by

$$\begin{aligned}
&\mathbf{a}\,\mathbf{b} \\
&= (a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k})(b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}) \\
&= -\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b}
\end{aligned}$$

• now use bilinearity
to compute the product of two general quaternions
viz

$$\begin{aligned}
&(\alpha + \mathbf{a})(\beta + \mathbf{b}) \\
&= \alpha\beta + \alpha\,\mathbf{b} + \mathbf{a}\,\beta + \mathbf{a}\,\mathbf{b} \\
&= \alpha\beta - \mathbf{a}\cdot\mathbf{b} + \beta\,\mathbf{a} + \alpha\,\mathbf{b} + \mathbf{a}\times\mathbf{b}
\end{aligned}$$

• define

the conjugate $\bar{q}$ of a quaternion $q = \alpha + \mathbf{a}$

as

$\bar{q} = \alpha - \mathbf{a} =$ a quaternion

• define

the norm Nq of a quaternion q

as

$Nq = q\bar{q} =$ a nonnegative scalar

• then

the multiplicative inverse q^{-1} of a nonzero quaternion q

is their quotient

viz

$$q^{-1} = \frac{\bar{q}}{Nq}$$

• the discovery/invention of quaternions
in 1843 by Hamilton
was a notable point
in the development/history of mathematics

• the quaternions contain the complex numbers
& thus show that
number systems do not end with the complex numbers

• the quaternions show the need to study
noncommutative operations

• the quaternions opened up vast new areas
for algebraic study

• the quaternions show the need to consider
higher dimensional spaces

• the quaternions have
substantial geometric & physical interpretations & uses
just as the real numbers & the complex numbers have

• the 8-element set

$$\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$$

is a nonabelian multiplicative group of order 8
which is called
the quaternion group

• the multiplication table
for the three nonunity basis quaternions

$$\mathbf{i}, \mathbf{j}, \mathbf{k}$$

is summarized by the equations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$
$$\mathbf{i}\,\mathbf{j} = \mathbf{k} \;\&\; \mathbf{j}\mathbf{i} = -\mathbf{k}$$
$$\mathbf{j}\mathbf{k} = \mathbf{i} \;\&\; \mathbf{k}\,\mathbf{j} = -\mathbf{i}$$
$$\mathbf{k}\,\mathbf{i} = \mathbf{j} \;\&\; \mathbf{i}\,\mathbf{k} = -\mathbf{j}$$

which are determined by the single iterated equality

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\mathbf{k} = -1$$

• quaternions were discovered/invented
by Hamilton in 1843;
for about a dozen years earlier Hamilton had thought of
how to define satisfactorily the product of two vectors;
finally on October 16, 1843,
while he was walking with his wife
along the Royal Canal outside of Dublin,
the idea of quaternions
occurred to him in a flash of inspiration;
in Hamilton's own words
'Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely

$$i^2 = j^2 = k^2 = ijk = -1$$

which contains the Solution of the Problem, but of course, as an inscription, has long since mouldered away.'

• quaternion
is a good English word
& has been around since the 14th century;
it means
a set of four parts/persons/things
& was chosen by Hamilton
to name his newly found objects

• etymology

quaternion (English)
from
quaternio (Latin) = quaternion
from
quaterni (Latin) = four each
from
quater (Latin) = four times
from
quattuor (Latin) = four

• bioline
William Rowan Hamilton
1805 - 1865
Irish
algebraist, analyst, astronomer, physicist, linguist

☐ the Q & O multiplicative triplets
aka
quaternion & octonion multiplications made easy

• the quaternion number system $\mathbb{H}$
is by definition & a little proof
a 4-dimensional real normed conjugated
noncommutative associative
linear division algebra
with bilinear multiplication
&
with three basic unit quaternions (besides unity)

$$i,\ j,\ k$$

whose products satisfy the condition:
the ordered triple $(i,\ j,\ k)$ is a cyclic system
viz

$$i^2 = -1$$

$$j^2 = -1$$

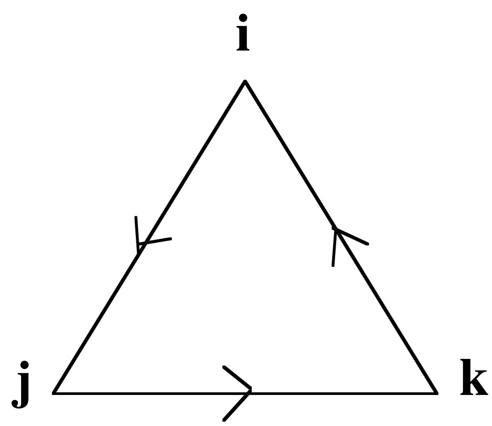
$$k^2 = -1$$

$$ij = k \ \&\ ji = -k$$

$$jk = i \ \&\ kj = -i$$

$$ki = j \ \&\ ik = -j$$

which has the following geometric mnemonic
for the six product equations
based on an equilateral triangle
with the three units at the vertices
and the sides directed coherently



which simply specifies the cyclic order of
i, j, k
viz
$\langle i, j, k\rangle = \langle j, k, i\rangle = \langle k, i, j\rangle$
the equations of a cyclic system
depending only on the cyclic order of the triple;
a reinforcement from the diagram occurs
in noting that
the plus sign in a product equation corresponds to
the positive = counterclockwise direction
around the triangle
&
the negative sign in a product equation corresponds to
the negative = clockwise direction
around the triangle

• the octonion number system $\mathbb{O}$
is by definition & a little proof
an 8-dimensional real normed conjugated
noncommutative nonassociative
linear algebra
with bilinear multiplication
&
with seven basic unit octonions (besides unity)

$e_n \quad (n \in \underline{7})$

s t
each of the following seven ordered triples
is a cyclic system:

$e_1 \quad e_2 \quad e_4$

$e_1 \quad e_3 \quad e_7$

$e_1 \quad e_5 \quad e_6$

$e_2 \quad e_3 \quad e_5$

$e_2 \quad e_6 \quad e_7$

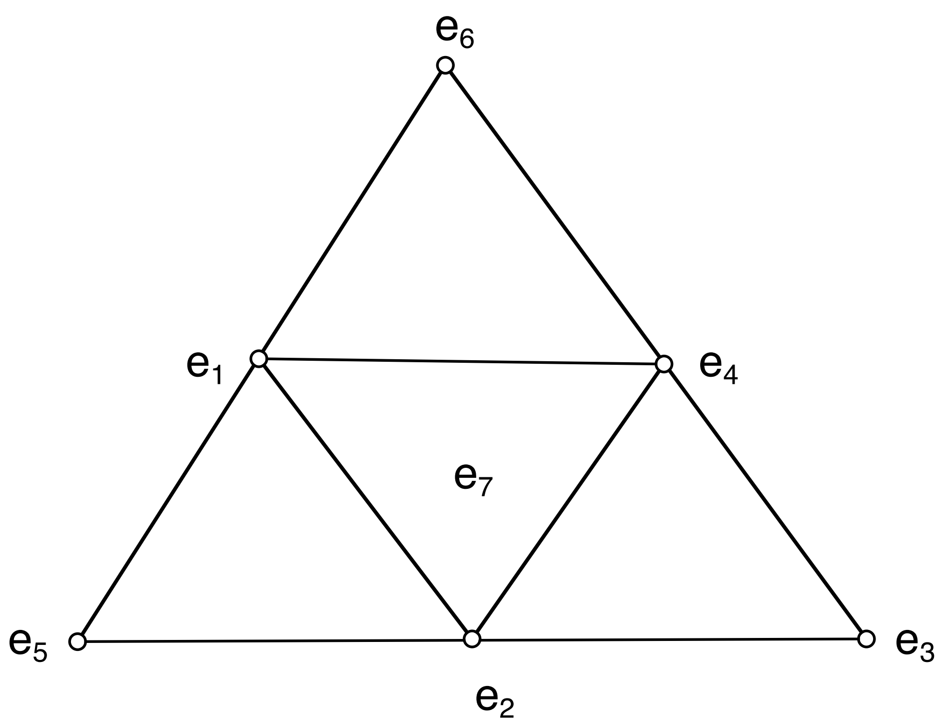
$e_3 \quad e_4 \quad e_6$

$e_4 \quad e_5 \quad e_7$

starting with any of the above triples
and repeatedly adding 1 to the subscripts mod 7
will yield all triples in the given cyclic order

GG20-15

a geometric mnemonic
for the above seven cyclic systems
is based on an equilateral triangle
as shown below;
the seven basic nonunity octonions
are distributed at
the three vertices,
the centroid,
the three side-midpoints
as indicated on the diagram;
there are seven 'lines'
viz
the three sides,
the three medians,
the curvilinear midpoint triangle;
think of the sides of the original triangle
and the curvilinear midpoint triangle
as oriented positively= in the counterclockwise direction;
think of the three medians as directed
from vertex to centroid to opposite side-midpoint;
each pair of units lies on just one line
and this line contains just one other unit and thus
the diagram determines a unique cyclic order
of these three units;
the seven cyclic systems
may now be readily read off the diagram



GG20-17

• the above diagram
is a picture of the Fano plane
viz
a finite projective plane with
7 points & 7 lines,
3 points on a line & 3 lines on a point

• bioline
Gino Fano
1871-1952
Italian
geometer

• the word octonion
was formed out of analogy with the word quaternion
using the Latin word octo = eight
which comes from the Greek word οκτω = eight

• the first published account of octonions
occurred in 1845 in a paper by
Arthur Cayley
1821-1895
English
algebraist, geometer, applied mathematician, lawyer,
third most prolific mathematican of all time
(Euler is first, Cauchy is second);
an earlier unpublished discovery/invention
of octonions occurred in 1843 in the work of
John T Graves,
an English mathematican
and college friend of Hamilton
with whom he corresponded

□ in extending
the real numbers to the complex numbers,
the complex numbers to the quaternions,
the quaternions to the octonions,
the primary difficulty is in defining multiplication
(as Hamilton first found out
in the instance of trying to multiply 3-vectors);
however there is a unified procedure to do so
ie to extend multiplication in a uniform fashion
in these three cases;
think of a complex number
as an ordered pair of real numbers,
think of a quaternion
as an ordered pair of complex numbers,
think of an octonion
as an ordered pair of quaternions;
at each of these three steps
multiplication can be defined by
the single formula

$$(a,b)(c,d) = (ac - d\bar{b},\ \bar{a}d + cb)$$

☐ some general info about algebras

D. linear algebras

let

• F $\in$ field

then

• a linear algebra over F

$=_{ab}$ an algebra over F

$=_{df}$ a a finite dimensional vector space A over F

provided with

a bilinear binary operation in A

wic

multiplication in A

• a division algebra over F

$=_{df}$ an algebra over F

wi

zero - divisorless

T. there are exactly four
real normed division algebras
viz
the real numbers $\mathbb{R}$ of dim 1
the complex numbers $\mathbb{C}$ of dim 2
the quaternions $\mathbb{H}$ of dim 4
the octonions $\mathbb{O}$ of dim 8
&
there are exactly four
real conjugated division algebras
viz
the real numbers $\mathbb{R}$ of dim 1
the complex numbers $\mathbb{C}$ of dim 2
the quaternions $\mathbb{H}$ of dim 4
the octonions $\mathbb{O}$ of dim 8