The Sigma Summation Notation

#18 of Gottschalk's Gestalts

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N. files

• the file of length n ($n \in \mathbb{P}$)

$= $ the n - file

$=_{dn}$ $\underline{n}$

$=_{rd}$ n file

$=_{df}$ the set of the first n positive integers

$= \{1, 2, \cdots, n\}$

$= \mathbb{P}[1, n]$

• the file of length zero

$= $ the 0 - file

$=_{dn}$ $\underline{0}$

$=_{rd}$ zero file

$=_{df}$ the empty set $\varnothing$

• thus

$\underline{0} = \varnothing$

$\underline{1} = \{1\}$

$\underline{2} = \{1, 2\}$

$\underline{3} = \{1, 2, 3\}$

etc

N. the basic capital sigma summation notation

• the sum

$a_1 + a_2 + \cdots + a_n$

$=_{rd}$ a_1 plus a_2 plus $\cdots$ plus a_n

of elements $a_1, a_2, \cdots, a_n$ ($n \in \mathbb{P}$) of an additive semigroup

= the (termwise) sum of the ordered n - tuple $(a_1, a_2, \cdots, a_n)$

= the (termwise) sum of the $\underline{n}$ – family $(a_i | i \in \underline{n})$

$=_{dn} \sum (a_1, a_2, \cdots, a_n)$

$=_{rd}$ summation / sum of $(a_1, a_2, \cdots, a_n)$

$=_{dn} \sum (a_i | i \in \underline{n})$

$=_{rd}$ summation / sum of $(a_i | i \in \underline{n})$

$$=_{dn} \sum_{i \in \underline{n}} a_i$$

$=_{rd}$ summation / sum for i in $\underline{n}$ of a_i

$$=_{dn} \sum_{i=1}^{n} a_i$$

$=_{rd}$ summation / sum from i equals 1 to n of a_i

wh $i \in$ var $\underline{n}$ or at least i is an integer - valued variable

that contains $\underline{n}$ in its range

N. in the sigma summation notation

$$\sum_{i=1}^{n} a_i$$

i is a bound = dummy = umbral variable,

i is called ' the summation index (variable)' ,

the set of values of i used in the summation (viz $\underline{n}$ here)

is called ' the summation range' ,

a_i ($i \in \underline{n}$) is called ' the ith summand' or ' the ith addend' ;

the summation index variable is often taken to be i

because i is the initial letter of ' index' and of ' integer' ;

summation index variables

are frequently taken from the alphabetic run h, i, j, k

and other good candidates are m, n, r

C. in 1755 Euler introduced capital sigma Σ

to denote continued sums;

the indicial notation was added later by others;

the Greek letter sigma Σ σ

corresponds phonetically & in transliteration to

the Latin – English letter ess S s

wi the initial letter of summa (Latin) = sum GG18 - 5

N. extended capital sigma summation notation

• $\sum_{i=m}^{n} a_i$

$=_{rd}$ summation / sum from i equals m to n of a_i

$=_{df}$ $a_m + a_{m+1} + \cdots + a_n$

wh

$m, n \in \mathbb{Z}$ st $m \leq n$

&

$i \in \text{var}\ \mathbb{Z}[m,\ n]$

or

i is a variable

whose values are integers

and

whose range includes $\mathbb{Z}[m,\ n]$

&

$a_m,\ a_{m+1}, \cdots,\ a_n$

are elements of an additive semigroup

• the (termwise) sum of $(a_i \mid i \in I)$

$=_{dn} \sum (a_i \mid i \in I)$

$=_{rd}$ summation / sum of $(a_i \mid i \in I)$

$=_{dn} \sum_{i \in I} a_i$

$=_{rd}$ summation / sum for i in I of a_i

$=_{df} \sum_{j \in \underline{n}} a_{\varphi j}$

wh

$(a_i \mid i \in I)$ is a nonempty finite family

in an additive commutative semigroup

&

$n = \text{crd } I$

&

$\varphi\colon \underline{n} \to I \in$ bijection

note: if I is also a totally ordered set,

then φ is uniquely definable as an order - isomorphism

&

the commutativity of the semigroup may be dropped

• the (elementwise) sum of A

$=_{dn} \ \sum A$

$=_{rd}$ summation / sum of A

$=_{df} \ \sum (a \mid a \in A) = \sum_{a \in A} a$

wh

A is a nonempty finite subset

of an additive commutative semigroup

N. the sigma summation notation for series

• the sum of the right series

$$a_m + a_{m+1} + \cdots = \sum_{i=m} a_i$$

$$=_{dn} \sum_{i=m}^{\infty} a_i$$

$=_{rd}$ summation / sum from i equals m to infinity of a_i

$$=_{df} \lim_{n \to \infty} \sum_{i=m}^{n} a_i \quad (n \in \text{var } \mathbb{Z}[m, \ \rightarrow)) \quad \text{iie}$$

wh

$m \in \mathbb{Z}$

&

$a_m, \ a_{m+1}, \ \cdots$

are elements of an additive topological semigroup

• the sum of the left series

$$\cdots + a_{n-1} + a_n = \sum^{i=n} a_i$$

$$=_{dn} \sum_{-\infty}^{i=n} a_i$$

$=_{rd}$ summation / sum from i equals n to minus infinity of a_i

$$=_{df} \lim_{m \to -\infty} \sum_{i=m}^{n} a_i \ \ (m \in \text{var}\ \mathbb{Z}(\leftarrow, n]) \ \text{iie}$$

wh

$n \in \mathbb{Z}$

&

$\cdots, a_{n-1}, a_n$

are elements of an additive topological semigroup

• the sum of the biseries

$$\cdots + a_{-1} + a_0 + a_1 + \cdots = \sum_{i \in \mathbb{Z}} a_i$$

$$=_{dn} \sum_{i=-\infty}^{i=\infty} a_i$$

$=_{rd}$ summation / sum from i equals minus infinity to i equals (plus) infinity

$$=_{df} \lim_{\substack{n \to \infty \\ m \to -\infty}} \sum_{i=m}^{n} a_i \quad (m, n \in \text{var}\, \mathbb{Z} \ \& \ m \le n) \ \text{iie}$$

wh

$\cdots,\ a_{-1},\ a_0,\ a_1,\ \cdots$

are elements of an additive topological semigroup

N. the sigma summation notation
with two or more summation indexes = indices;
here are three examples

- $$\sum_{i,j=1}^{n} a_{ij}$$

$=_{rd}$ summation / sum from i, j equal 1 to n of a_{ij}

= a sum of n^2 summands from a_{11} to a_{nn}

- $$\sum_{i,j,k=1}^{n} a_{ijk}$$

$=_{rd}$ summation / sum from i, j, k equal 1 to n of a_{ijk}

= a sum of n^3 summands from a_{111} to a_{nnn}

- $$\sum_{\substack{i,j=1 \\ i<j}}^{n} a_{ij}$$

$=_{rd}$ summation / sum from i, j equal 1 with i less than j to n of a_{ij}

= a sum of ${}_nC_2 = \frac{1}{2}n(n-1)$ summands from a_{12} to $a_{n-1,n}$

- it is understood that n is a positive integer & that the a' s are from an additive commutative semigroup GG18 - 12

Δ laws for elements a_i $(i \in \text{var}\,\mathbb{P})$
of any additive semigroup

• special range laws

$$\sum_{i=1}^{0} a_i = \sum_{i \in \varnothing} a_i = \sum \varnothing =_{df} 0 \quad \text{iie}$$

$$\sum_{i=1}^{1} a_i = a_1$$

$$\sum_{i=1}^{2} a_i = a_1 + a_2$$

$$\sum_{i=1}^{3} a_i = a_1 + a_2 + a_3$$

etc

Δ law for an element a
of any additive group

• the right distributive law for multiples

$$\left(\sum_{i=1}^{n}\alpha_i\right)a = \sum_{i=1}^{n}\alpha_i\, a \quad (n \in \mathbb{P} \;\&\; \alpha_i \in \mathbb{Z} \text{ for } i \in \underline{n})$$

Δ laws for elements a_i ($i \in \text{var}\,\mathbb{P}$)
of any additive commutative group ($n \in \mathbb{P}$)

• the negation law

$$-\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} -a_i$$

• the left distributive law for multiples

$$\alpha \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \alpha\, a_i \quad (\alpha \in \mathbb{Z})$$

Δ laws for elements a_i (i, j ∈ var ℙ) of any commutative ring (n ∈ ℙ)

• the first square - of - sum law

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i{}^2 + 2\sum_{\substack{i.j=1\\ i<j}}^{n} a_i a_j$$

• the second square - of - sum law

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (2-n)\sum_{i=1}^{n} a_i{}^2 + \sum_{\substack{i,j=1\\ i<j}}^{n} (a_i + a_j)^2$$

• the third square - of - sum law

$$\left(\sum_{i=1}^{n} a_i\right)^2 = n\sum_{i=1}^{n} a_i{}^2 - \sum_{\substack{i,j=1\\ i<j}}^{n} (a_i - a_j)^2$$

Δ law for elements $a_i,\ b_i$ ($i \in \text{var}\,\mathbb{P}$)
of any additive commutative semigroup ($n \in \mathbb{P}$)

• the additive law

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Δ laws for elements $a_i,\ b_i$ ($i \in \text{var}\,\mathbb{P}$)
of any additive commutative group ($n \in \mathbb{P}$)

• the subtractive law

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

• the linear law

$$\sum_{i=1}^{n}(\alpha a_i + \beta b_i) = \alpha \sum_{i=1}^{n} a_i + \beta \sum_{i=1}^{n} b_i \quad (\alpha, \beta \in \mathbb{Z})$$

Δ the binomial theorem

in any commutative ring ($n \in \mathbb{P}$)

• first form

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i\, b^{n-i}$$

• second form

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i}\, b^i$$

• third form

$$(a+b)^n = \sum_{\substack{i,j=0 \\ i+j=n}}^{n} \frac{n!}{i!\,j!} a^i\, b^j$$

note: the third form suggests

the trinomial theorem which has three indexes

& in general

the multinomial theorem which has say $r \geq 2$ indexes

D. the Bernoulli numbers $B_0, B_1, B_2, \cdots$

are definable by a generating function

$$\frac{x}{e^x - 1} \quad (x \in \text{real var})$$

as follows:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (\text{for } x \text{ near } 0)$$

whence

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30}$$

$$B_5 = 0,\ B_6 = \frac{1}{42},\ B_7 = 0,\ B_8 = -\frac{1}{30},\ B_9 = 0,$$

$$B_{10} = \frac{5}{66},\ \cdots$$

Δ the complex functions
exponential, sine, cosine
are definable as
everywhere - convergent power series
as follows:

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (\forall z \in \mathbb{C})$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \qquad (\forall z \in \mathbb{C})$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \qquad (\forall z \in \mathbb{C})$$

Δ an example;

the Laurent series of a complex function

that is analytic on the punctured plane

and

that has an essential singularity at the origin (z ∈ var ℂ)

$$\exp z \ + \ \exp \frac{1}{z}$$

$$= \ e^{z} + e^{\frac{1}{z}}$$

$$= \ 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

$$= \ \cdots + \frac{1}{3!z^3} + \frac{1}{2!z^2} + \frac{1}{1!z} + 2 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$= \ \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!z^n}$$

$$= \ \sum_{-\infty}^{n=0} \frac{z^n}{|n|!} \ + \ \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= \ \sum_{-\infty}^{n=-1} \frac{z^n}{|n|!} \ + 2 + \ \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

N. the capital pi production notation

$$\prod_{i=1}^{n} a_i = a_1 a_2 \cdots a_n$$

for the product of the elements $a_1,\ a_2, \cdots ,\ a_n \ \ (n \in \mathbb{P})$

of a multiplicative semigroup

is analogous to

the capital sigma summation notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

for the sum of the elements $a_1,\ a_2, \cdots ,\ a_n \ \ (n \in \mathbb{P})$

of an additive semigroup

eg

the factorial of $n \ \ (n \in \mathbb{P})$

$=$ factorial n

$=_{dn}$ $n!$

$=_{rd}$ n factorial $=$ n bang

$$=_{df} \prod_{r=1}^{n} r = \prod_{r=0}^{n-1} (n-r)$$

$= 1 \times 2 \times 3 \times \cdots \times n = n(n-1)(n-2)\cdots 1$

$=$ the product of the first n positive integers

note: just as the sum of the empty set
of additive semigroup elements
is defined to be the additive identity element zero iie
ie

$$\sum_{i \in \varnothing} a_i =_{df} 0 \quad \text{iie}$$

so analogously the product of the empty set
of multiplicative semigroup elements
is defined to be the multiplicative identity element unity iie
ie

$$\prod_{i \in \varnothing} a_i =_{df} 1 \quad \text{iie}$$

C. the notation n! for n factorial was introduced
in 1808 by Christian Kramp of Strasbourg, France;
in 1812 Gauss introduced capital pi Π
to denote continued products;
the indicial notation was added later by others;
the Greek letter pi Π π
corresponds phonetically & in transliteration to
the Latin - English letter pee P p
which is the initial letter of productum (Latin) = product

D. two dual kinds of factorial powers of x

where x is an element of a commutatve unital ring

& $n \in \mathbb{P}$

• the rising nth factorial power of x

$=_{dn}\ x^{\overline{n}}$

$=_{rd}$ x rising n (factorial)

$$=_{df} \prod_{r=0}^{n-1}(x+r)$$

$= x(x+1)(x+2)\cdots(x+n-1)$ which has exactly n factors

• the falling nth factorial power of x

$=_{dn}\ x^{\underline{n}}$

$=_{rd}$ x falling n (factorial)

$$=_{df} \prod_{r=0}^{n-1}(x-r)$$

$= x(x-1)(x-2)\cdots(x-n+1)$ which has exactly n factors

Δ the three binomial formulas / theorems
for three kinds of powers
in a commutative unital ring ($n \in \mathbb{P}$)

• the binomial formula / theorem
for ordinary powers

$$(a+b)^{n} = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}\, b^{r}$$

• the binomial formula / theorem
for rising factorial powers

$$(a+b)^{\overline{n}} = \sum_{r=0}^{n} \binom{n}{r} a^{\overline{n-r}}\, b^{\overline{r}}$$

• the binomial formula / theorem
for falling factorial powers

$$(a+b)^{\underline{n}} = \sum_{r=0}^{n} \binom{n}{r} a^{\underline{n-r}}\, b^{\underline{r}}$$

Δ three examples of infinite products

- $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}$

- $\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n)^2}\right) = \frac{2}{\pi}$

- $\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n+1)^2}\right) = \frac{\pi}{4}$

note: the first product
is the product of
the second product and the third product