What Is Geometry?

#17 of Gottschalk's Gestalts

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GG17-2

□ ¿ what is geometry ?

• ¿ what is geometry ?
or in just two words:
define geometry;
it appears that this is
a basic largely unanswered
question in mathematics

• it is to be remarked that the word 'geometry'
is ambiguous in that it can refer to
a particular mathematical object
or
many mathematical objects
or
the study of this object or these objects;
the words algebra/calculus/logic/topology
have a similar ambiguity

• a more suggestive and emphatic formulation
of the central question here
would be:
define Geometry with a capital gee

• it is easy enuf in general
to define a geometry with a little gee
ie
to define a particular geometry
eg

(1) plane euclidean geometry
is precisely definable by
a fairly complicated axiom system
which begins

a plane euclidean geometry
= df
a set E
equipped with
[structure described ito sets]
such that
[list of axioms expressed ito sets
with the lower predicate calculus
as logical/linguistic vehicle]

(2) plane projective geometry
is precisely definable by
a fairly simple axiom system
which begins in similar fashion

a plane projective geometry
= df
a set P
equipped with
[structure described ito sets]
such that
[list of axioms expressed ito sets
with the lower predicate calculus
as logical/linguistic vehicle]

• in each case (1) & (2) above
existence & isomorphic uniqueness theorems
can be proved

• note the modern insight into the nature of
an axiom system and an axiom
viz
an axiom system
= df
a definition
&
an axiom
= df
a clause of a definition;
it is likely that this clear insight became possible
only after the recognition of
set theory & symbolic logic
about a century ago

• note that the denotation of the phrase
'the set X equipped with the structure S'
is definable as a set
viz
the ordered pair (X, S)

• it is also easy
to define precisely
classes of geometries with a little gee;
here are three examples

• for each positive integer n
real n-dimensional euclidean geometry
= df
the study of
real n-dimensional euclidean space
which is defined to be
the real n-dimensional number set
equipped with the pythagorean distance function

• riemannian geometry
(a big part of differential geometry)
= df
the study of riemannian manifolds
which are defined to be differentiable manifolds
equipped with riemannian metrics
which in turn are defined locally
to be quadratic forms in the differentials
of the local coordinates

• algebraic geometry
= df
the study of algebraic varieties
where an algebraic variety
is defined to be the solution set
of a collection of algebraic equations

• now Geometry with a capital gee
could be 'defined' as follows:

Geometry with a capital gee
= df
the set of all geometries with a little gee

however
this only rephrases the question into

¿ what do the geometries with a little gee
have in common that would be
a determining characteristic
that justifies calling them 'geometries' ?

or more briefly

characterize the set of all geometries

• again
an unsatisfying answer would be of
an historical/sociological/operational nature
viz

a branch of mathematics is called a 'geometry'
provided that
a sufficiently large number of experts in that field
agree to call it a 'geometry'
or
to paraphrase briefly & humorously:
‘geometry’ is what geometers do
&
geometry = what geometers do

this is hardly a final definitive answer to the question
because it gives no hint as to the nature of
what we wish to call ‘geometry’;
and also we look for an answer
that is independent of
individual personal judgements/opinions
and that is independent of
the passage of time

• here are physiological/psychological attempts
to define Geometry ito the Geometer

(1) Geometry
= the study by the left brain
of the notions recognized by the right brain

(2) Geometry
= the responses by the left brain
to the questions posed by the right brain

(3) Geometry
= how the left brain perceives the right brain

(4) Geometry
= how the mind perceives the body

• the above statements presume
the usual hemispherical dichotomy
between the left brain and the right brain
viz
the left brain
uses symbols
and is analytic & logical & scientific & rational
while
the right brain
uses pictures
and is synthetic & intuitive & artistic & emotional

• the above statements may contain
a modicum of truth;
for example, they could provide some insight
into the historical origin of geometry;
but the primary difficulty in assessing these statements
is that it is not clear how to express these statements
in the language of the theory of sets
with the predicate calculus as logical/linguistic vehicle
(which is my sufficiency criterion
for something to be considered mathematics)

• a frequently offered candidate
for an answer to the question is

geometry
= df
the study of space

this attempted definition is
vague because ‘space’ is not here defined;
‘space’ cannot mean ‘physical space’ because
that leads to physics;
if ‘space’ means ‘topological space’,
then the assertion is that geometry = topology;
I would judge this ‘definition’ as inadequate
but nonetheless suggestive

• it appears that for every geometry
there is an associated object
that should be called a space
and is in particular a topological space;
thus a geometry needs a space to support it
but the same space can support
many different kinds of geometries;
the supporting space may be
locally euclidean ie a manifold

• 'topology' is precisely definable as follows

topology
= df
the study of topological spaces

since 'topological space' is precisely definable;
the notion of topological space captures precisely
the general notion of limit
or equivalently
the general notion of nearness

again
topology
=df
the study of the general notion of limit

• from an historical point of view
it appears that the basic notions of topology
grew out of both geometry and analysis

• so in my opinion
the best I can offer
toward the definition of a geometry
is that it is a topological space
with an additional structure
but precisely what kind of structure
is not given to my understanding;
one can only offer many examples
and say these are all called geometries

• for all the reasons above it is considered by many
that it is best to regard
geometry/topology (one entity)
as one of the first or principal subdivisions/branches
of mathematics,
the other two principal subdivisions/branches
of mathematics
being algebra
(=the study of finitary relations & operations)
and
analysis
(=the study of limit properties of numbers
& functions of numbers);
to be sure
geometry/topology
can be split further,
but in which any problems of classification
would appear to be
technically rather than essentially difficult

• out of despair of ever answering the question
¿ what is geometry ?
simply say

a geometry
= df
anything you want to call a geometry
that has
a topological space in it
&
geometry can be thot of as containing topology
if necessary

• it may be that the question
will be answered automatically and easily
as time goes by;
just by general implicit, even unaware agreement,
the answer may arrive some day

- the etymology of the word ‘geometry’

geometry
↑
géométrie (French)
= geometry
↑
geometria (Latin)
= geometry
↑
γεωμετρια (Greek)
= geometry
↑
γεω – (Greek)
= earth
+
–μετρια (Greek)
= measuring of
↑
γη (Greek)
= earth
+
μετρον (Greek)
= measure

• historical origin of the word ‘geometry’

the ancient Egyptians developed
practical geometric/surveying procedures
to restore ownership of land
covered by the annual flooding of the Nile river;
the ancient Greeks were aware of this;
whence the name
geometry = earth-measure

• Geometry is the Measure of the World

☐ a Riemannian insight on the nature of geometry

it is likely that

Georg Friedrich Bernhard Riemann
1826-1866
German
analyst, geometer, number theorist,
topologist, physicist

was the first to clearly recognize
that there is
a profound philosophical/mathematical distinction
between
the notion of space & the notion of geometry
and that
the same basic underlying space
can have/support many geometries

in brief oracular declaration

• Topology is the platform of Space;
Space is built on Topology

• Space is the platform of Geometry;
Geometry is built on Space

• Algebra & Analysis
are builders
of a structure Space
on the platform Topology

• Algebra & Analysis
are builders
of a structure Geometry
on a platform Space

in terms of modern concept-building

• start with Topology as topological space

• add Euclidean Topology as local topology
get
topological manifold

• add Analytic Geometry as coordinate system
get
coordinate manifold

• add Analysis as differentiability
get
differentiable manifold

• add Geometry as metric/connection/etc
get
geometric manifold

□ Klein's Erlangen Program

• the German noun
das Programm
means both
program/plan
&
inaugural address/lecture
which was delivered, according to custom,
to the faculty of a German university
by a new appointee in order to show off
the appointee's scholarly prowess
& thus justify the appointment
as teacher & researcher

• the phrase
das Erlanger Programm (German)
= the Erlangen Program
thus has two meanings;
the phrase refers to
the inaugural address by the German mathematician
Felix Klein
1849-1925
delivered in 1872
to the faculty of the University of Erlangen, Germany,
upon his appointment there;
the phrase also refers to
the mathematical content of his inaugural address
viz
a certain way of viewing geometries as described below;
note that ‘Erlanger’ is the adjective form
of the German noun ‘Erlangen’,
the name of the city in south-central Germany
where the University of Erlangen is located

• Klein's Erlangen Program
= df
the characterization/classification/definition
of geometries according to the transformation groups
that leave invariant their characteristic features,
the geometries for which this is possible
thus being called 'Kleinian geometries'
as described more fully below

• a Kleinian geometry
= df
the study of predicates = properties & relations
that are invariant under a given group of transformations
acting on a given space
=
the theory of invariants of a transformation group
=
the invariant theory of a transformation group

• not all geometries should be called Kleinian
eg
any geometry
whose automorphismn group
reduces to the identity map
can hardly be considered to be Kleinian

• one would expect the orbit of any point
under the automorphism group of a Kleinian geometry
to be the entire space

• bioline

Christian Felix Klein

(full name usually shortened to Felix Klein)

1849-1925

German

algebraist, analyst, geometer, topologist, educator, historian of mathematics, mathematical physicist

☐ some
philosophical & poetical & theological
quotes on geometry

• Thomas Browne:
God is like a skilful Geometrician.

• Samuel Butler:
For he by geometric scale,
Could take the size of pots of ale.

• Albrecht Dürer:
... Geometry, without which no one
can either be or become an absolute artist.

• Euclid:
There is no royal road to geometry.
[Said to Pharaoh Ptolemy I of Egypt.]

• Gustave Flaubert:
Poetry is a subject as precise as geometry.

• Thomas Hobbes:
In Geometry (which is the only science that it hath pleased God hitherto to bestow on mankind) men begin at settling the significations of their words; which ... they call Definitions.

• Edna St. Vincent Millay:
Euclid alone has looked on beauty bare.

• Christopher Morley:
O basic and everlasting geometry.

• Plato:
God continually geometrizes.

• Plato:
Let no one ignorant of geometry enter here.
[Said to be an inscription over the gateway
to Plato's Academy of Athens.]