Integers Defined As Sets

#14 of Gottschalk's Gestalts

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by Walter Gottschalk

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500 Angell St #414
Providence RI 02906

GG14-2

□ integers defined as sets

• the von Neumann description of an ordinal
as the set of all smaller ordinals
leads to the following recursive definition
of the nonnegative integers as sets

• the successor function
on the class of sets to the class of sets
is defined explicitly
as follows

$n \mapsto n^{+} = n \cup \{n\} \quad (n \in \text{set})$

where n^{+} is called the successor of n
and the notation n^{+} (read n plus) is suggested by $n + 1$

• the nonnegative integers
are defined as sets
by the following recursive definition

rec def

$0 = \varnothing$

$n^+ = n \cup \{n\}$ $(n \in$ set var$)$

• more fully

let

$0 \ =_{rd} \text{ zero} =_{df} \varnothing$

$1 \ =_{rd} \text{ one} =_{df} 0^{+} = 0 \cup \{0\} = \{0\} = \{\varnothing\}$

$2 \ =_{rd} \text{ two} =_{df} 1^{+} = 1 \cup \{1\} = \{0,1\} = \{\varnothing, \{\varnothing\}\}$

$3 \ =_{rd} \text{ three} =_{df} 2^{+} = 2 \cup \{2\} = \{0,1,2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

etc

according to the standard

Indo – Arabic decimal positional notation and terminology;

note that

each nonnegative integer is defined as

the set of all smaller nonnegative integers;

also note that

each nonnegative integer is defined ito

the empty set and the set - builder

which suggests the Latin motto

Omnia ex nihilo. = Everything from nothing.

• this recursive definition
of the nonnegative integers as sets
may be complemented by
the following explicit definition
of the negative integers as sets

$- n$
$=_{rd}$ minus n
$=_{df}$ $\{n\}$ (n $\in$ nonzero nonneg int)

note that the ' minus sign' here is just a part of the notation;
we have not yet defined
the unary operation of negation of integers;
in case of notational ambiguity,
use the elbow ¬ here in place of the dash –

• we can now construct
the ladder of integers as sets:

etc

$4 = \{0, 1, 2, 3\}$

$3 = \{0, 1, 2\}$

$2 = \{0, 1\}$

$1 = \{0\}$

$0 = \varnothing$

$-1 = \{1\}$

$-2 = \{2\}$

$-3 = \{3\}$

$-4 = \{4\}$

etc

• thus we have the following
five basic sets of integers
in which everything is defined as a set:

(1) $\mathbb{P}$
= ' open cap pe'
= ' pe'
= the set of all positive integers
= the set of positive integers
= the positive integer set
= the positive integers
= $\{1,\ 2,\ 3,\ \cdots\}$

(2) $\overline{\mathbb{P}}$
= ' open cap pe bar'
= ' pe bar'
= the set of all negative integers
= the set of negative integers
= the negative integer set
= the negative integers
= $\{-1,\ -2,\ -3,\ \cdots\}$

(3) $\mathbb{N}$

= ' open cap en'

= ' en'

= the set of all nonnegative integers

= the set of nonnegative integers

= the nonnegative integer set

= the nonnegative integers

= $\{0,\ 1,\ 2,\ 3,\ \cdots\}$

(4) $\overline{\mathbb{N}}$

= ' open cap en bar'

= ' en bar'

= the set of all nonpositive integers

= the set of nonpositive integers

= the nonpositive integer set

= the nonpositive integers

= $\{0,\ -1,\ -2,\ -3,\ \cdots\}$

(5) $\mathbb{Z}$

= ' open cap zee'

= ' zee'

= the set of all integers

= the set of integers

= the integer set

= the integers

= $\{\cdots, \ -3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3, \ \cdots\}$

• note that the ' minus sign' in $\overline{\mathbb{P}}$ and $\overline{\mathbb{N}}$ are conveniently placed above the letters and not before; as for the ' etymology' of the open capital letters,

$\mathbb{P}$ is from positive

$\mathbb{N}$ is from nonnegative, natural, number

$\mathbb{Z}$ is from die Zahl (German) = number;

the open type style,

which is a style seldom appearing in mathematical literature,

was adopted for the sake of this particular usage

and for immediate recognition;

these letters are now in almost universal use

with the present meaning

• the relation of order between integers,
the unary operation of absolute value formation for integers, and
the unary operation of negation of integers
are simply and explicitly definable as follows:

• order

$m < n \Leftrightarrow m \in n$

$-m < n \qquad (m \neq 0)$

$-m < -n \Leftrightarrow n \in m \qquad (m \neq 0 \neq n)$

wh $m, n \in$ nonneg int

• absolute value

$|n| = n$

$|-n| = n \qquad (n \neq 0)$

wh $n \in$ nonneg int

• negation

$-0 = 0$

$-n = \neg n \qquad (n \neq 0)$

$-(\neg n) = n \qquad (n \neq 0)$

wh $n \in$ nonneg int

• the customary recursive definitions
of addition and multiplication
for nonnegative integers are:

rec def of addition for nonnegative integers

$m + 0 = m$

$m + n^{+} = (m + n)^{+}$

wh m, n $\in$ nonneg int var

rec def of multiplication for nonnegative integers

$m \cdot 0 = 0$

$m \cdot n^{+} = (m \cdot n) + m$

wh m, n $\in$ nonneg int var

• the binary operations of
addition and multiplication for integers
are definable by cases
eg
$(-m) + (-n) = -(m + n)$
&
$(-m)(-n) = mn$
wh $m, n \in$ nonneg int

• the binary operation of
subtraction for integers
is definable explicitly
ito addition and negation for integers
viz
$m - n = m + (-n)$

• the partial binary operation of division for integers
occupies a special position
since division by zero is not defined
(nor even sensibly definable)
and since two integers (quotient and remainder)
result from the operation of division for integers;
division for integers is ring division;
division
for rational numbers,
for real numbers, and
for complex numbers
in which only a single number, the quotient, results
is field division

• division for integers

q is the (integer) quotient
and
r is the (nonnegative integer) remainder
for the division of
the dividend m
divided by
the divisor $n \neq 0$

means by definition

$$m = nq + r \quad (0 \leq r < |n|)$$

wh $m, n, q, r \in$ int;
for this definition to be meaningful,
a unique existence theorem for q and r must be proved