An Esthetic/Philosophical Appreciation of the Biggest Little Formula in Mathematics:

e to the πi power plus 1 equals 0

#11 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
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by Walter Gottschalk

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☐ one of the most pleasing little formulas in mathematics
and one with almost mystical overtones is

Euler's epic epiphany equality

$e^{\pi i} + 1 = 0$

'ee to the pi eye plus one equals zero'

epiphany
= pr ee-PIF-uh-nee
= df a sudden insightful manifestation/perception/revelation

the first three letters of the words
epic & epiphany
can be discerned in the first three symbols
$e^{\pi i}$
of the formula
$e^{\pi i} + 1 = 0$
since p is the English equivalent of the Greek π

□ this little formula

$$e^{\pi i} + 1 = 0$$

in spite of its short length,
is one of the most remarkably
comprehensive and representative formulas
in mathematics

the formula
$e^{\pi i} + 1 = 0$
beautifully and dramatically displays in brief compass

- the unity of mathematics

&

- the interdependence of the branches of mathematics

consider the following facts about the formula
$e^{\pi i} + 1 = 0$

□ the formula $e^{\pi i} + 1 = 0$
first appeared in pioneering work by
Leonhard Euler
= one of the great mathematicians of all time

Euler = pr OI-ler

the formula $e^{\pi i} + 1 = 0$
was first published in 1748, two & a half centuries ago

of the three letters e, i, π in the formula $e^{\pi i} + 1 = 0$
Euler introduced the letters e and i in their present sense
and popularized the letter π in its present sense

thus the formula $e^{\pi i} + 1 = 0$ represents
• the history of mathematics

□ bioline
Euler, Leonhard
1707-1783
Swiss; spent many years in Germany and Russia
algebraist, analyst, geometer, number theorist, probabilist,
applied mathematician, calculating prodigy;
the most prolific mathematician of all time;
the first modern mathematical universalist

☐ the formula

$$e^{\pi i} + 1 = 0$$

is a special case of the general Euler formula

$$e^{iz} = \cos z + i \sin z$$

which holds for all complex numbers z

to specialize
the more general formula
$e^{iz} = \cos z + i \sin z$
to
the more special formula
$e^{\pi i} + 1 = 0$
take $z = \pi$

thus the formula $e^{\pi i} + 1 = 0$ represents
the fundamental mathematical processes of
• generalization & specialization

□ the formula $e^{\pi i} + 1 = 0$ contains
the most important & the most frequently occurring relation in mathematics viz
• equality =

□ the formula $e^{\pi i} + 1 = 0$ contains
the most important & the most frequently occurring operation in mathematics viz
• addition +

□ the formula $e^{\pi i} + 1 = 0$ contains

the five most important & most frequently occurring numbers in mathematics viz
• zero = 0, one = 1, eye = i, ee = e, pi = π

the most important integers
• 0, 1

the most important imaginary number
• i

the most important transcendental numbers
• e, π

□ the formula $e^{\pi i} + 1 = 0$ represents
the three principal branches of mathematcis viz
• algebra, analysis, geometry/topology
the various ways in which this representaion occurs
are spelled out below

□ in the formula $e^{\pi i} + 1 = 0$ the items

the additive identity	0
the multiplicative identity	1
the imaginary unit	i
the equality relation	$=$
the addition operation	$+$
multiplication in the juxtaposition	πi
exponentiation in the superscription	$e^{\pi i}$

are basic notions of
algebra
=df the study of finitary operations & relations

thus the formula $e^{\pi i} + 1 = 0$
represents
• algebra

the items 0, 1, =, +, multiplication, exponentiation
are basic notions of
arithmetic
= df the study of the simpler computations involving
the four fundamental binary operations
addition, subtraction, multiplication, division
applied to integers & rational numbers mainly
&
the elementary theory of numbers
= df the study of integers without the use of limit processes
ie without analysis
which may be considered to be subbranches of algebra

thus the formula $e^{\pi i} + 1 = 0$
represents
• arithmetic
&
• the elementary theory of numbers

the imaginary unit i
along with
0, 1, e, π, =, +, multiplication, exponentiation
are basic notions in the complex number system C
which is of fundamental importance in
algebra, analysis, geometry/topology

thus the formula $e^{\pi i} + 1 = 0$
represents
• the complex number system

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□ the formula $e^{\pi i} + 1 = 0$
contains two transcendental numbers e and π

e
= the base of natural logarithms
= the natural logarithm base
= the nat log base
$= \lim_{h \to 0} (1+h)^{\frac{1}{h}}$ wh h ∈ pos real var

$= \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k$ wh k ∈ pos real var

$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ wh n ∈ pos int var

$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

= the unique positive real number x such that $\int_1^x \frac{1}{t} dt = 1$

the natural exponential function
e^x
has e as base

the natural logarithm function
$\log_e x$
has e as base

π
= the circle ratio
= the ratio of circumference to diameter
for all euclidean circles

$$= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$= \ 4\int_0^1 \sqrt{1 - x^2}\, dx$$

both e and π
are definable ito the limit processes of
analysis
= df the study of limit properties of
numbers & functions of numbers;
inp, the study of
differentiation & integration

thus the formula $e^{\pi i} + 1 = 0$
represents
• analysis

☐ the value of e to 12 decimal places is
e = 2.71828 18284 59+

¿ why 12 places ?

because the 13th place is occupied by the first 0

there is a nice east-to-remember
but apparently accidental pattern
to the first 15 decimal places of e
viz
e = 2.7
    1828   1828
    45   90   45
    +

the value of π to 31 decimal places is
π = 3.14159 26535 89793 23846 26433 83279 5+

¿ why 31 places ?

because the 32nd place is occupied by the first 0

a mnemonic for π to 31 places, up to the first zero, is to count the letters of words in the paragraph
How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics! All of thy geometry, Herr Planck, is fairly hard. You too struggle? Yes, we acquire knowledge daily.
the first sentence is due to Sir James Hopwood Jeans

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□ e has been calculated to more than
50 million decimal places (as of 1998)

π has been calculated to more than
206 billion decimal places (as of 1999)

the calculation of a large number of decimal places
of e and π may be used as a test
for the accuracy/efficiency/speed of a computer

calculation/computation in general
and of e and π in particular
is based on algorithms

computer science
= df the study of algorithms

thus the formula $e^{\pi i} + 1 = 0$
represents
• computer science

□ the circle ratio π
has a geometric reference

thus the formula $e^{\pi i} + 1 = 0$
represents
• geometry
= df the study of space
[this definition (?) is not satisfactory
because it is not clear how this so-called definition
picks out those axiom systems
that we would like to call geometries;
no satisfactory definition of Geometry with a capital gee
has ever been offered to my knowledge
altho each geometry with a little gee is readily definable
by the appropriate axiom system]

□ the explanation of the formula $e^{\pi i} + 1 = 0$
leads to the exposition of much and varied mathematics

thus the formula $e^{\pi i} + 1 = 0$
represents
• mathematics education
= df the study of how to study mathematics
ie how to teach/learn/write/read mathematics

□ the formula $e^{\pi i} + 1 = 0$ contains
7 = a nice prime number
of explicit individual symbols
0, 1, i, e, π, =, +
in single occurrence
for the 7 most important mathematical objects
and also 2 more implicit contextual symbols
juxtaposition for multiplication
&
superscription for exponentiation
for good measure

thus the formula $e^{\pi i} + 1 = 0$
contains/represents
• beautiful & efficient notation

□ the formula $e^{\pi i} + 1 = 0$
is to be certainly regarded as
beautiful mathematics

here are some philosophical questions

(1) ¿ is beauty a necessary or a sufficient condition
for mathematical/physical truth ?

(2) more generally
John Keats (1795-1821, English poet)
proclaimed
$(B = T) = K$
where
B = beauty
T = truth
K = knowledge
in the following passage of his poem
'Ode on a Grecian Urn' (1819)
'Beauty is truth, truth beauty,'- that is all
Ye know on earth, and all ye need to know.
¿ agree or disagree ?

(3) consider the four-category classification

	beautiful	useful
I	yes	yes
II	yes	no
III	no	yes
IV	no	no

¿ is there some mathematics in each category ?

¿ how about good mathematics ?

¿ the best ?

(4) ¿ is the formula $e^{\pi i} + 1 = 0$ useful ?

□ the formula $e^{\pi i} + 1 = 0$
has suggested thoughts as above on

the unity of mathematics

the diversity of mathematics

the beauty of mathematics

the utility of mathematics

the simplicity of mathematics

the complexity of mathematics

the mystery of mathematics

¡ they are all -y words !

thus the formula $e^{\pi i} + 1 = 0$
represents
• the philosophy of mathematics

¡ note that 'philosophy' is a -y word too!

☐ It could be argued that the notion of set, represented by the elementhood relation symbol $\in$, is the most fundamental and the most important notion of mathematics, and that the theory of sets is not suggested by the formula $e^{\pi i} + 1 = 0$. But, on the other hand, from an historical perspective, the pervasiveness of set theory in mathematics is only about a century old. Actually, every mathematical object is definable as a set. For example,

0 = df $\varnothing$

1 = df {0} = {$\varnothing$}

For that matter, it could be argued that the empty set $\varnothing = 0$ is the most important set in mathematics. If not the empty set $\varnothing$, what else? This is an application of the philosophical Principle of Sufficent Reason.

thus the formula $e^{\pi i} + 1 = 0$
represents
• the theory of sets

□ Other possible gaps in the universality
of the formula $e^{\pi i} + 1 = 0$ may be pointed out. For example:

The formula $e^{\pi i} + 1 = 0$ contains only constants and no variables altho variables are used in definitions of objects named in the formula.

The formula $e^{\pi i} + 1 = 0$ contains no general function sign such as $f(x)$ altho the particular binary functions of addition, multiplication, exponentiation occur in the formula.

The formula $e^{\pi i} + 1 = 0$ contains no derivative or integral altho e and π are defined by limit processes, and derivative and integral are also defined by limit processes. The numbers e and π may be characterized elegantly in calculus using derivatives/ integrals.

The formula $e^{\pi i} + 1 = 0$ contains no symbol from logic such as that for a propositional operation or a quantifier
altho equality = is often thought of as a logical notion, and logic can be considered to be a special subbranch of algebra.

The formula $e^{\pi i} + 1 = 0$ contains no explicit reference to topology altho geometry could be stretched a bit to encompass topology or vice versa depending on how you like to stretch a point. Topology could be briefly defined as the study of the abstract notion of limit, and the formula suggests limit processes in e and π.

☐ It is clear that the formula $e^{\pi i} + 1 = 0$
represents mathematics ie
pure mathematics = core mathematics
in many ways. How, if at all, does the formula $e^{\pi i} + 1 = 0$ represent applied mathematics beyond the suggested algorithmic computation of e and π? Of course, the individual symbols in the formula $e^{\pi i} + 1 = 0$ occur thruout mathematics, pure and applied. Here are two examples of particularly interesting occurrences of π, one occurrence in the theory of probability and the other occurrence in physics.

The number π can occur in the theory of probability
eg π occurs in
the Buffon needle problem/theorem/experiment.
Let a homogeneous thin straight needle of length L units be dropped at random on a flat horizontal table that is evenly scored by parallel straight lines consecutively separated by D units where $D \geq L$. Then it can be calculated/observed that the probability of the needle falling on a line is

$$\frac{2L}{\pi D}$$

The number π can occur in particle physics
eg π occurs in
the Heisenberg indeterminacy/uncertainty principle/relation

$$\Delta x \times \Delta p \geq \frac{h}{4\pi}$$

where

x = position

p = momentum

h = Planck's constant = the (elementary) quantum of action
= the ratio of the energy of a photon to its frequency

Δ = the error in the measurement of

☐ The ultimate formula/symbol for everything is yet to be found, no doubt. Does it exist? Does it lurk in the formula $e^{\pi i} + 1 = 0$? How about the elementhood sign $\in$? It could be argued that

$$\in = \{(x, y) : x \in y\}$$

where x and y are set variables. If any mathematical statement can be reduced in principle to statements of the form $x \in y$ that are combined by the use of logical symbols say those from the lower predicate calculus, then it appears that the class $\in$ contains all mathematical knowledge. The big problem is how to access it.

☐ a mathematical palindrome

Q. ¿ what is your favorite transcendental number ?

A. i prefer pi

thus the formula $e^{\pi i} + 1 = 0$
represents
• mathematical humor

□ a strictly mathematical note

to prove Euler's formula

$$e^{iz} = \cos z + i \sin z \qquad (\forall z \in \mathbb{C})$$

appeal to the three everywhere-convergent power series expansions

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (\forall z \in \mathbb{C})$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \qquad (\forall z \in \mathbb{C})$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \qquad (\forall z \in \mathbb{C})$$

these series could be regarded as
theorems for a real variable
and
definitions for a complex variable

□ curiouser and curiouser formulas

the phrase 'curiouser and curiouser'
was said by Alice in the book
'Alice's Adventures in Wonderland' (1865)
which was written by Lewis Carroll
= a mathematican with the mundane name of
Charles Lutwidge Dodgson
1832-1898
English
writer, mathematics don at Oxford University

$$i^i = e^{-\frac{\pi}{2}} \quad \text{(pv)}$$

Georgie Porgie said 'Hi!
The principal ith power of i
Is the number e to
Minus π over 2
But I cannot begin to tell why.'

$$\sqrt[i]{i} = e^{\frac{\pi}{2}} \quad \text{(pv)}$$

Georgie Porgie said 'Hi!
The principal ith root of i
Is the number e to
Plus π over 2
But I cannot begin to tell why.'

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each Georgie Porgie formula
may be transformed into the other
by taking the minus one power
of each side of the equation

the last line of the limericks is ambiguous

if Georgie Porgie does know the mathematical proof
but feels it is too complicated to be said briefly say,
then the following discussion
is what he could tell

if Georgie Porgie does not know the mathematical proof,
then the following discussion
is what he could be told

N. real & complex variables

$x, y, u, v \in$ real var

$z, w \in$ complex var

$x + iy = z$

$u + iv = w$

note that u, v, w, x, y, z are the last six letters of the alphabet in alphabetic order

D. the absolute value of a complex number

the absolute value of z

= dn $|z|$

= rd absolute z = z absolute

= df $\sqrt{x^2 + y^2} \geq 0$

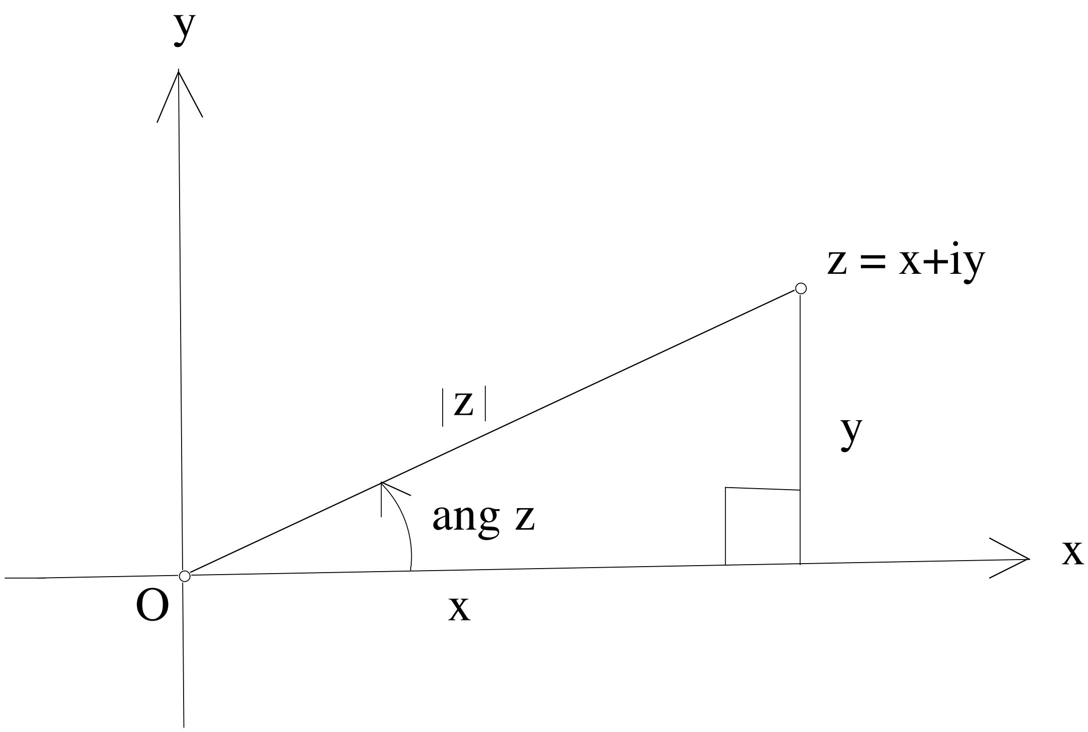
D. the angle of a nonzero complex number

the angle of z

= dn ang z

= rd angle z

= df $\tan^{-1} \frac{y}{x} \in Qz$



geometric picture & interpretation
of the absolute value $|z|$
&
of the angle ang z
of a complex number
$z = x+iy$

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D. complex logarithm

w is the (complex) (natural) logarithm of z (to the base e) $(z \neq 0)$

$= \text{df } z = e^{w}$

$= e^{u+iv}$

$= e^{u}e^{iv}$

$= e^{u}(\cos v + i \sin v)$

$= e^{u} \cos v + ie^{u} \sin v$

&

therefore

$$\begin{cases} x = e^{u} \cos v \\ y = e^{u} \sin v \end{cases}$$

$$\begin{cases} u = \ln\sqrt{x^{2} + y^{2}} \\ v = \tan^{-1} \dfrac{y}{x} \in Q(x,\ y) \end{cases}$$

$\log z = \ln|z| + i \operatorname{ang} z$

D. complex power

the complex power with base z and with exponent w

= the w power of z

= dn z^w

= rd z to the w (power)

= df $e^{w \log z}$

D. complex radical / root

the complex radical / root with radicand z and with index w

= the w root of z

= dn $\sqrt[w]{z}$

= rd w root (of) z

= df $z^{\frac{1}{w}}$

R. the little Euler formula
& the two Georgie Porgie formulas

$$e^{\pi i} + 1 = 0$$

$$i^{i} = e^{-\frac{\pi}{2}}$$

$$\sqrt[i]{i} = e^{\frac{\pi}{2}}$$

are all three readily derivable from each other
and so
they are pairwise equivalent in that sense

direct derivations of the Georgie Porgie formulas
follow below

$$|i| = 1$$

$$\text{ang}\, i = \frac{\pi}{2} + 2\pi n \qquad (n \in \text{int})$$

$$\log i \;=\; \ln|i| + i\,\text{ang}\, i \;=\; \ln 1 + i\left(\frac{\pi}{2} + 2\pi n\right) \;=\; i\left(\frac{\pi}{2} + 2\pi n\right)$$

$$i^{i} \;=\; e^{i \log i} \;=\; e^{-\frac{\pi}{2} + 2\pi n}$$

$$\therefore\; i^{i} = e^{-\frac{\pi}{2}} \qquad (\text{pv})$$

$$\sqrt[i]{i} \;=\; i^{\frac{1}{i}} \;=\; i^{-i} \;=\; e^{-i \log i} \;=\; e^{\frac{\pi}{2} + 2\pi n}$$

$$\therefore\; \sqrt[i]{i} \;=\; e^{\frac{\pi}{2}} \qquad (\text{pv})$$

pv = df principal value

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□ coda
a complimentary complement
a superlative supplement
a summery summation
an endearing endpoint
a ding-a-ling Ding an sich (German) = lit: thing-in-itself

$$e^{\pi i} + 1 = 0$$

'Ee to the pie eye plus won
Goes poof' is a benison
For it wraps up a lot
In a very small spot
And proves math is always great fun.