Theme & Variations: Defining a Group

#100 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2003

GG100-1 (26)

500 Angell St #414
Providence RI 02906

GG100-2

C. the notion of group,
which is basic and of central importance
in mathematics,
is here defined
with full detail
in 19 equivalent ways;
commentary abounds

D1. groups

a group

$=_{df}$ an ordered quadruple

$G = (G, e, *, \circ)$

st

the following four axioms are satisfied:

(0) generic axiom

(01) $G \in$ set

wh $G =_{cl}$ the prop of G

(02) $e \in G$

wh $e =_{cl}$ the identity (element) of G

(03) $*: G \to G \in$ unary operation in G

$x \mapsto x*$

wh $* =_{rd}$ star $=_{cl}$ the stellation of G

& $x* =_{rd}$ x star $=_{cl}$ the stellate of x

(04) $\circ: G \times G \to G \in$ binary operation in G

$(x, y) \mapsto x \circ y$

wh $\circ =_{rd}$ oh / op $=_{cl}$ the composition of G

& $x \circ y =_{rd}$ x oh / op y $=_{cl}$ the composite of x and y

(1) bilateral identity axiom for composition

• e $\in$ bilateral identity for composition

$=_{df}$ $x \circ e = x = e \circ x \quad (x \in G)$

(2) bilateral inverse axiom for composition

• stellation $\in$ bilateral inversion for composition

$=_{df}$ $x \circ x^* = e = x^* \circ x \quad (x \in G)$

(3) associative axiom for composition

• composition $\in$ associative

$=_{df}$ $(x \circ y) \circ z = x \circ (y \circ z) \quad (x, y, z \in G)$

D2. groups

obtain D2 from D1 by changing D1 as follows:
in axioms (1) & (2) of D1
replace the word ' bilateral' by the word ' right'
& omit the right hand parts of the equations

D3. groups

obtain D3 from D1 by changing D1 as follows:
in axioms (1) & (2) of D1
replace the word ' bilateral' by the word ' left'
& omit the left hand parts of the equations

D4. groups

a group

$=_{df}$ an ordered pair

☝ $= (G, \circ)$

st

the following three axioms are satisfied:

(0) generic axiom

(01) $G \in$ set

wh $G =_{cl}$ the prop of $\mathcal{G}$

(02) $\circ : G \times G \to G \in$ binary operation in G

$(x, y) \mapsto x \circ y$

wh $\circ =_{rd}$ oh / op $=_{cl}$ the composition of $\mathcal{G}$

& $x \circ y =_{rd}$ x oh / op y $=_{cl}$ the composite of x and y

(1) associative axiom for composition

• composition $\in$ associative

$=_{df}$ $(x \circ y) \circ z = x \circ (y \circ z) \quad (x, y, z \in G)$

(2) bilateral identity - inverse existential axiom for composition

• composition has a bilateral identity st each element has a bilateral inverse

$=_{df}$ $\exists e \in G . (\forall x \in G . x \circ e = x = e \circ x)$

$\& (\forall x \in G . \exists x^* \in G . x \circ x^* = e = x^* \circ x)$

D5. groups

obtain D5 from D4 by changing D4 as follows:
in axiom (2) of D4
replace the word ' bilateral' by the word ' right'
& omit the right hand parts of the equations

D6. groups

obtain D6 from D4 by changing D4 as follows:
in axiom (2) of D4
replace the word ' bilateral' by the word ' left'
& omit the left hand parts of the equations

D7. groups

obtain D7 from D4 by changing D4 as follows:

replace the line in D4
(01) G $\in$ set
by the line
(01) G $\in$ nonempty set

replace axiom (2) in D4 by the axiom
(2) bilateral solubility - existential axiom
for composition
• simple composite equations in one unknown $\in$ soluble
$=_{df}$ $\forall a, b \in G . \exists x, y \in G . a \circ x = b \,\&\, y \circ a = b$

D8 / 9 / 10. groups

a group

$=_{df}$ an associative groupoid

with bilateral / right / left identity element

st every element is bilaterally / right / left invertible

D11 / 12 / 13. groups

a group

$=_{df}$ a semigroup

with bilateral / right / left identity element

st every element is bilaterally / right / left invertible

D14 / 15 / 16. groups

a group

$=_{df}$ a bilateral / right / left monoid

st every element is bilaterally / right / left invertible

D17. groups

a group

$=_{df}$ a nonempty semigroup

st all simple equations in one unknown are soluble

D18. additive groups

an additive group

$=_{df}$ an ordered quadruple

$\mathcal{G} = (G, 0, -, +)$

st

the following four axioms are satisfied:

(0) generic axiom

(01) $G \in$ set

wh G $=_{cl}$ the prop of $\mathcal{G}$

(02) $0 \in G$

wh 0 $=_{rd}$ oh / zero $=_{cl}$ the zero (element) of $\mathcal{G}$

(03) $-: G \to G \in$ unary operation in G

$$x \mapsto -x$$

wh $-$ $=_{rd}$ minus $=_{cl}$ the negation of $\mathcal{G}$

& $-x$ $=_{rd}$ minus x $=_{cl}$ the negate of x

(04) $+: G \times G \to G \in$ binary operation in G

$$(x, y) \mapsto x + y$$

wh $+$ $=_{rd}$ plus $=_{cl}$ the addition of $\mathcal{G}$

& $x + y$ $=_{rd}$ x plus y $=_{cl}$ the sum of x and y

(1) bilateral identity axiom for addition

• $0 \in$ bilateral identity for addition

$=_{df}$ $x + 0 = x = 0 + x \quad (x \in G)$

(2) bilateral inverse axiom for addition

• negation $\in$ bilateral inversion for addition

$=_{df}$ $x + (-x) = 0 = (-x) + x \quad (x \in G)$

(3) associative axiom for addition

• addition $\in$ associative

$=_{df}$ $(x + y) + z = x + (y + z) \quad (x, y, z \in G)$

D19. multiplicative groups

a multiplicative group

$=_{df}$ an ordered quadruple

$\mathcal{G} = \left(G,\ 1,\ ^{-1},\ \cdot\right)$

st

the following four axioms are satisfied:

(0) generic axiom

(01) $G \in$ set

wh $G =_{cl}$ the prop of $\mathcal{G}$

(02) $1 \in G$

wh $1 =_{rd}$ one / unity $=_{cl}$ the unity (element) of $\mathcal{G}$

(03) $^{-1} : G \to G \in$ unary operation in G

$$x \mapsto x^{-1}$$

wh $^{-1} =_{rd}$ recip / inverse

$=_{cl}$ the reciprocation / inversion of $\mathcal{G}$

& $x^{-1} =_{rd}$ x recip / inverse

$=_{cl}$ the reciprocal / inverse of x

(04) $\cdot : G \times G \to G \in$ binary operation in G

$$(x, y) \mapsto x \cdot y$$

wh $\cdot =_{rd}$ times $=_{cl}$ the multiplication of $\mathcal{G}$

& $x \cdot y =_{rd}$ x times y $=_{cl}$ the product of x and y

(1) bilateral identity axiom for multiplication

• $1 \in$ bilateral identity for multiplication

$=_{df} \; x \cdot 1 = x = 1 \cdot x \quad (x \in G)$

(2) bilateral inverse axiom for multiplication

• reciprocation $\in$ bilateral inversion for multiplication

$=_{df} \; x \cdot x^{-1} = 1 = x^{-1} \cdot x \quad (x \in G)$

(3) associative axiom for multiplication

• multiplication $\in$ associative

$=_{df} \; (x \cdot y) \cdot z = x \cdot (y \cdot z) \quad (x, y, z \in G)$

C. some comments,
mathematical and semiphilosophcal,
about the definitions of a group

- each of the following five triples
D1 / 2 / 3
D4 / 5 / 6
D8 / 9 / 10
D11 / 12 / 13
D14 / 15 / 16
differ in regard to
bilateral / right / left identities
matched by
bilateral / right / left inverses;
it turns out that these variations
all amount to the same thing

- D1 / 2 / 3

contain no existential quantifiers,
only universal quantifiers
& have four items in the given structure
& have four axioms

- D4 / 5 / 6 / 7

contain existential quantifiers
as well as universal quantifiers
& have two items in the given structure
& have three axioms

- D7 is likely the shortest definition of a group

- D8 thru 17

are all verbally stated definitions
that take less rich algebraic structures
as previousy defined;
the terms of the sequence
groupoid, semigroup, monoid, group
have increasingly richer structure

• additive groups and mutiplicative groups
D18 / 19
simply refer to the notation and the terminology,
not to the notions or the generality;
D18 / 19
are variants in notation and terminology
of the ' general' definition D1
with ' general' notation and terminology;
D18 / 19 bring the notions
to words and symbols that are more familiar;
however there is no distinction in generality
among D1 / 18 / 19;
in practice additive groups are usually used
only for abelian groups;
multiplicative groups are often used
with e instead of 1 denoting the identity element;
usually in a multiplicative group
juxtaposition xy is used to denote
the product $x \cdot y$
of the elements x and y

• a group is a nonempty set provided with
a nullary operation
& a unary operation
& a binary operation
subject to certain conditions

• the notion of group
is the mathematical embodiment
of the intuitive notion of symmetry;
the notion of ordered set
is the mathematcial embodiment
of the intuitive notion of ranking;
the notion of topological space
is the mathematical embodiment
of the intuitive notion of nearness;
together these three notions of
group, ordered set, topological space,
with their
specializations & generalizations & combinations,
cover the mathematical landscape

C. generic axiom

• the generic axiom is suggested as
the first axiom of an axiom system;
it specifies the kind ie 'genus' of each object
named as part of the original structure
but says no more;
it replaces the closure axiom / axioms
of an earlier version of axiom systems

• the word 'prop' has the sense of
base, basis, carrier, foundation, support;
a new word seemed in order

C. words & symbols

• the word in various languages

group (English)

= le groupe (French)

= die Gruppe (German)

= il gruppo (Italian)

= el grupo (Spanish)

• the word group (groupe)
was first used by Galois in 1830

• $\mathcal{G}$, G $\leftarrow$ <u>g</u>roup

• e $\leftarrow$ the German noun die <u>E</u>inheit = oneness, unity

• Cayley introduced the righthand superscript -1
for the inverse in 1844

C. the abstract notion of group
took a long time to crystalize;
first considerations
that turned out to be related to groups
were devoted to
the study of permutations of finite sets
eg the roots of a polynomial equation;
involved were
Lagrange (ca 1770)
Galois (ca 1830) (particularly)
Cayley (ca 1844)
& others;
a clear simple statement of the abstract group axioms
took many mathematicians
and most of the 19th century to develop
and was successfully realized in the early 20th century