The Fundamental Theorem of Calculus

#3 of Gottschalk's Gestalts

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☐ The Fundamental Theorem of Calculus

• The Fundamental Theorem of Calculus = FTC
is the centerpiece jewel of calculus;
it expresses the focally located fact that
the two basic operations of calculus,
viz differentiation and integration,
are inverse operations
ie each undoes what the other does

• below there are listed
various versions of various forms of FTC;
they all say the same thing
and are essentially equivalent in the sense that
each version of each form is easily proved
from any other version of any other form

• to better understand FTC,
this basic theorem,
this overwhelmingly important message,
it is helpful to say it in many ways
and to be both precise & concise

• it is to be assumed that

the Riemann integral

$$\int_a^b f(x)dx$$

has been defined and its existence proved

where

x is a real variable,

f(x) is a real - valued continuous function

on the bounded closed real interval

$$[a,\ b] = \{x : a \le x \le b\}$$

for real numbers a and b with $a \le b$,

and therefore

$$\int_a^b f(x)dx$$

is a real number

□ (1) a word & symbol version of the upper - lim form of FTC

FTC. Let $f:[a,b] \to \mathbb{R}$ be continuous where $a, b \in \mathbb{R}$ with $a < b$. Then

$$\exists \frac{d}{dx} \int_a^x f(t)dt = f(x) \quad (a \leq x \leq b)$$

where x and t are real variables.

□ (2) a word & symbol version of the lower - lim form of FTC

FTC. Let $f:[a,b] \to \mathbb{R}$ be continuous where $a, b \in \mathbb{R}$ with $a < b$. Then

$$\exists \frac{d}{dx} \int_x^b f(t)dt = -f(x) \quad (a \leq x \leq b)$$

where x and t are real variables.

☐ (3) an all - word version
of the upper - lim form of FTC

FTC. The derivative with respect to
the variable upper lim of the Riemann integral
of a continuous real - valued function
on a plural bounded closed real interval
exists and equals the integrand
evaluated at the upper lim.

☐ (4) an all - word version
of the lower - lim form of FTC

FTC. The derivative with respect to
the variable lower lim of the Riemann integral
of a continuous real - valued function
on a plural bounded closed real interval
exists and equals the negative of the integrand
evaluated at the lower lim.

□ (5) an all - symbol version
of the upper - lim form of FTC

FTC. $a, b \in \mathbb{R}$ st $a < b$
& $f:[a,b] \to \mathbb{R} \in$ cont
$\Rightarrow$

$$\exists \frac{d}{dx}\int_a^x f(t)dt = f(x) \quad (a \le x \le b)$$

wh $x, t \in$ var $\mathbb{R}$

□ (6) an all - symbol version
of the lower - lim form of FTC

FTC. $a, b \in \mathbb{R}$ st $a < b$
& $f:[a,b] \to \mathbb{R} \in$ cont
$\Rightarrow$

$$\exists \frac{d}{dx}\int_x^b f(t)dt = -f(x) \quad (a \le x \le b)$$

wh $x, t \in$ var $\mathbb{R}$

☐ (7) a word & symbol version of the antiderivative form of FTC

FTC. Let $f:[a,b]\to\mathbb{R}$ be continuous where $a,b\in\mathbb{R}$ with $a<b$. Then

• $F:[a,b]\to\mathbb{R}$ st $\exists\,\frac{d}{dx}F(x)=f(x)\ \ (a\le x\le b)$

$\Rightarrow$

$\int_a^b f(x)dx = F(b)-F(a)$

• $\exists F:[a,b]\to\mathbb{R}$ st $\exists\,\frac{d}{dx}F(x)=f(x)\ \ (a\le x\le b)$

where x is a real variable.

☐ (8) an all - word version
of the antiderivative form of FTC

FTC. For a continuous real - valued function
on a plural bounded closed real interval,
an antiderivative exists and
the Riemann integral equals
the value of any antiderivative at the upper lim
minus
the value of the antiderivative at the lower lim.

□ (9) an all-symbol version
of the antiderivative form of FTC

FTC. $a, b \in \mathbb{R}$ st $a < b$
& $f:[a,b] \to \mathbb{R} \in$ cont
$\Rightarrow$

• $F:[a,b] \to \mathbb{R}$ st $\exists \frac{d}{dx} F(x) = f(x) \quad (a \le x \le b)$

$\Rightarrow$

$\int_a^b f(x)dx = F(b) - F(a)$

• $\exists F:[a,b] \to \mathbb{R}$ st $\exists \frac{d}{dx} F(x) = f(x) \quad (a \le x \le b)$

wh $x \in$ var $\mathbb{R}$

□ comments on FTC

• the upper lim and lower lim forms 1 thru 6 of FTC above
treat the endpoints separately & analogously
but the antiderivative forms 7, 8, 9 of FTC above
treat the endpoints together & as symmetrically as possible

• let us think of the big ideas
& ignore the small details;
denote the Riemann integral by RI;
call the antiderivative difference
the Newton - Leibniz integral & denote it by NLI;
then the shortest form of FTC is
FTC. RI = NLI
¿cryptic, eh wot?

• FTC is a catch - 22 because

FTC says that

in order to evaluate the Riemann integral

$$\int_a^b f(x)dx$$

one has to find a function

$F(x) \quad (a \leq x \leq b)$

whose derivative is

$f(x) \quad (a \leq x \leq b)$

¿ but what is this function F(x) ?

it is the Riemann integral

$$\int_a^x f(t)dt \quad (a \leq x \leq b)$$

¿ what is the resolution of this paradox?

we know

F(x)

= the antiderivative of f(x)

= the primitive of f(x)

= the indefinite integral of f(x)

$= \int f(x)dx$

of many a function f(x)

without the consideration

of the Riemann integral of f(x)

because

every time we have a differentiation formula

$$\frac{d}{dx}F(x) = f(x)$$

we have an indefinite integration formula

$$\int f(x)dx = F(x) + C$$

& conversely;

in general, differentiation formulas are easy to come by

but integration formulas ab initio are hard to come by

□ The Weak Fundamental Theorem of Calculus = WFTC

• WFTC. Let $F:[a,b] \to \mathbb{R}$ be continuously differentiable where $a,b \in \mathbb{R}$ with $a < b$. Then

$$\int_a^b \frac{d}{dx}F(x)dx = F(b) - F(a)$$

and again

$$\int_{x=a}^{x=b} dF(x) = F(b) - F(a)$$

• WFTC is an immediate consequence of FTC

☐ synoptic paraphrases of FTC & WFTC

• FTC & WFTC together say that
differentiation = forming the derivative
and
integration = forming the integral
are inverse operations

• FTC says

(0) take a continuous function

(1) first integrate

(2) then differentiate

(3) and you get the original function back;

in symbols

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

• WFTC says

(0) take a continuously differentiable function

(1) first differentiate

(2) then integrate

(3) and you get the original function back;

in symbols

$$\int_a^b \frac{d}{dx} F(x)dx = F(b) - F(a)$$

☐ comparing / contrasting the RI & the NLI

• a RI is usually difficult to evaluate precisely
from its definition

• a RI may often be evaluated approximately
from its definition or
by a more - or - less similar process

• a NLI is often easier to evaluate precisely
than the corresponding RI

• both the RI & the NLI
are important
conceptually & computationally
but in different ways

• FTC is a mathematical miracle;
FTC says that this is
the best of all possible mathematical worlds
(here there are overtones
of Leibniz' s philosophy
that this is the best of all possible worlds
&
of Voltaire' s satire ' Candide'
on that philosophy)
because we do not have to choose between
the important RI & the important NLI
in that they are equal in value (pun intended)

☐ GI of FTC

where

GI

= geometric interpretation

= geometric image / imagery

= geometric imagination

= geometric intuition

• think of a positive continuous function

$y = f(x) \quad (a \leq x \leq b)$

& its graph = a curve

• think of $A(x)$ as the area

under the curve

above the x - axis

to the right of the ordinate at a

to the left of the ordinate at x

• GI of RI is

in words

RI equals area under curve

&

in symbols

$$\int_a^x y\,dx = A(x)$$

$$\int_a^b f(x)\,dx = A(b)$$

• GI of FTC is

in words

rate of change wrt abscissa

of area under curve

equals ordinate

&

in symbols

$$\frac{dA}{dx} = y$$

☐ comment on terminology

Since derivatives and integrals,
as well as other good things
such as sums of series (Let the good things roll!),
are all limits, one could say that the notion of limit
(involving numbers and functions of numbers)
is the fundamental notion of analysis.
The word 'analysis' is a fancy word for 'calculus'.
Well, not exactly. The customary distinction is that
calculus is the more-or-less
'elementary' beginning part of analysis,
that is, the part that is learned first.

The word ' limit'
in its most widely used sense (as above)
is too important to be used also
in the sense of ' bound / boundary' .
Hence, say
' lower & upper limb $=_{ab}$ lim'
for a & b in the definite integral $\int_a^b f(x)dx$.
Observe limb & lim
are both pronounced the same.
The word limb has
an appropriate astronomical significance
viz an extreme edge of the apparent disk
of a celestial body.
Of course, limit $=_{ab}$ lim .