The Transcendental Trellis

#1 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

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GG1-2

Δ the transcendental trellis

- the transcendental trellis

is a pictorial representation of defining relations
among the 26 basic transcendental functions:
the exponential function
the six basic trigonometric functions
the six basic hyperbolic functions
and their 13 inverses viz
the logarithmic function
the six basic inverse trigonometric functions
the six basic inverse hyperbolic functions

these 26 functions are clearly
the most important transcendental functions
in elementary analysis ie calculus

the trellis is based upon
Euler' s formula
$e^{ix} = \cos x + i \sin x$
where x may be considered to be a real or complex variable
GG1 - 3

• the transcendental trellis is conveniently printed on
five $8\frac{1}{2}$ by 11 inch sheets;
to display the transcendental trellis
arrange these five sheets on a flat surface
in the form of an aitch:
the square in the middle
6 trig on the upper left
6 trig^{-1} on the lower left
6 hyp on the upper right
6 hyp^{-1} on the lower right
with the edges of the sheets in contact
(see GG1 - 25 on; display sheets are not imprinted
with their numbers)

• the transcendental trellis is intended to be
a unifying simplifying conceptual device (gestalt)
in the understanding of the elementary transcendental functions;
once it is seen and studied
a breadth of oversight and a depth of insight
are hopefully acquired
that are imprinted indelibly in the memory

Δ notation used in the trellis
& concepts / facts suggested by the trellis
are described below

- exp

= the exponential function
= exp x
= e^x

- 6 trig

= the canonical list
of the six basic trigonometric functions
expressed in exponential form
& ito sin and cos

- 6 hyp

= the canonical list
of the six basic hyperbolic functions
expressed in exponential form
& ito sinh and cosh

- log

= the logarithm function

= log x

- 6 $trig^{-1}$

= the canonical list

of the six basic inverse trigonometric functions

expressed in logarithmic form

- 6 hyp^{-1}

= the canonical list

of the six basic inverse hyperbolic functions

expressed in logarithmic form

- side - to - side horizontal motion

= $\leftrightarrow$

= are analogous to each other

- up - and - down vertical motion

= I

= an elongated capital letter eye
from the initial letter of ' inverse'

= are inverse functions

- outward radial motion

= radial arrows

= permits the definition / expression / formulation of

• ' i in' refers to the explicit presence
of the letter denoting the imaginary unit i in
(1) Euler' s formula & companion formula
(2) the exponential expressions
for the six trigonometric functions
(3) the logarithmic expressions
for the six inverse trigonometric functions

• ' i out' refers to the explicit absence
of the letter denoting the imaginary unit i in
(1) Lambert' s formula & companion formula
(2) the exponential expressions
for the six hyperbolic functions
(3) the logarithmic expressions
for the six inverse hyperbolic functions

• the exponential formulas
6 trig
for the six trigonometric functions
are essentially equivalent to

Euler' s formula
$e^{ix} = \cos x + i \sin x$

and the equivalent

Euler' s companion formula
$e^{-ix} = \cos x - i \sin x$

note that replacing x by $-x$ and using parity
interchanges the latter two formulas

the equivalences just mentioned are suggested
in the trellis by vertical equivalence signs

• the exponential formulas
6 hyp
for the six hyperbolic functions
are essentially equivalent to

Lambert' s formula
$e^{x} = \cosh x + \sinh x$

and the equivalent

Lambert' s companion formula
$e^{-x} = \cosh x - \sinh x$

note that replacing x by $-x$ and using parity
interchanges the latter two formulas

the equivalences just mentioned are suggested
in the trellis by vertical equivalence signs

• parity

20 of the 26 functions in the trellis have parity as indicated below

$\sin x \rightarrow \text{odd} \leftarrow \sinh x$

$\cos x \rightarrow \text{even} \leftarrow \cosh x$

$\tan x \rightarrow \text{odd} \leftarrow \tanh x$

$\cot x \rightarrow \text{odd} \leftarrow \coth x$

$\sec x \rightarrow \text{even} \leftarrow \text{sech}\, x$

$\csc x \rightarrow \text{odd} \leftarrow \text{csch}\, x$

$\sin^{-1} x \rightarrow \text{odd} \leftarrow \sinh^{-1} x$

$\cos^{-1} x \rightarrow \text{none} \leftarrow \cosh^{-1} x$

$\tan^{-1} x \rightarrow \text{odd} \leftarrow \tanh^{-1} x$

$\cot^{-1} x \rightarrow \text{odd} \leftarrow \coth^{-1} x$

$\sec^{-1} x \rightarrow \text{none} \leftarrow \text{sech}^{-1} x$

$\csc^{-1} x \rightarrow \text{odd} \leftarrow \text{csch}^{-1} x$

e^x has no parity

$\log x$ has no parity

GG1 - 11

• periodicity

13 of the 26 functions in the trellis are periodic;
the periods of these 13 functions are given below

$\sin x \to 2\pi$	$2\pi i \leftarrow \sinh x$
$\cos x \to 2\pi$	$2\pi i \leftarrow \cosh x$
$\tan x \to \pi$	$\pi i \leftarrow \tanh x$
$\cot x \to \pi$	$\pi i \leftarrow \coth x$
$\sec x \to 2\pi$	$2\pi i \leftarrow \operatorname{sech} x$
$\csc x \to 2\pi$	$2\pi i \leftarrow \operatorname{csch} x$

none of the 12 inverse functions is periodic

e^x is periodic with period $2\pi i$

$\log x$ is not periodic

• reciprocation & inversion = turning upside down

reciprocating the members of the equations in 6 trig
& using the reciprocal trigonometric identities,
the column in 6 trig is inverted

reciprocating the members of the equations in 6 hyp
& using the reciprocal hyperbolic identities,
the column in 6 hyp is inverted

replacing x by $\frac{1}{x}$ in the equations in 6 trig^{-1}
& using the reciprocal trigonometric identities,
the column in 6 trig^{-1} is inverted

replacing x by $\frac{1}{x}$ in the equations in 6 hyp^{-1}
& using the reciprocal hyperbolic identities,
the column in 6 hyp^{-1} is inverted

• to pass syntactically = formally in the trellis
from Euler' s formula & companion formula
to Lambert' s formula & companion formula
delete i & annex h to cos and sin

• to pass syntactically = formally in the trellis
from the formulas in 6 trig
to the formulas in 6 hyp
delete i & annex h to the functions' abbreviations

• a syntactic = formal passage in the trellis
from the formulas in 6 trig^{-1}
to the formulas in 6 hyp^{-1}
not only deletes i & annexes h
but also requires other changes

- to pass semantically = algebraically in the trellis

from Euler' s formula & companion formula

to Lamberts' formula & companion formula

replace x by $\frac{x}{i} = -ix$

& use the parity formulas

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

& use the conversion formulas

$\sin ix = i \sinh x$

$\cos ix = \cosh x$

- to pass semantically = algebraically in the trellis

from Lambert' s formula & companion formula

to Euler' s formula & companion formula

replace x by ix

& use the conversion formulas

$\sinh ix = i \sin x$

$\cosh ix = \cos x$

- to pass semantically = algebraically in the trellis

from		to	use the identity
$\sin x$	$\rightarrow$	$\cos x$	$\cos x = \sqrt{1 - \sin^2 x}$
$\cos x$	$\rightarrow$	$\sin x$	$\sin x = \sqrt{1 - \cos^2 x}$
$\tan x$	$\rightarrow$	$\sec x$	$\sec x = \sqrt{1 + \tan^2 x}$
$\cot x$	$\rightarrow$	$\csc x$	$\csc x = \sqrt{1 + \cot^2 x}$
$\sec x$	$\rightarrow$	$\tan x$	$\tan x = \sqrt{\sec^2 x - 1}$
$\csc x$	$\rightarrow$	$\cot x$	$\cot x = \sqrt{\csc^2 x - 1}$

• to pass semantically = algebraically in the trellis

from		to	use the identity
$\sinh x$	$\rightarrow$	$\cosh x$	$\cosh x = \sqrt{1+\sinh^2 x}$
$\cosh x$	$\rightarrow$	$\sinh x$	$\sinh x = \sqrt{\cosh^2 x - 1}$
$\tanh x$	$\rightarrow$	$\operatorname{sech} x$	$\operatorname{sech} x = \sqrt{1-\tanh^2 x}$
$\coth x$	$\rightarrow$	$\operatorname{csch} x$	$\operatorname{csch} x = \sqrt{\coth^2 x - 1}$
$\operatorname{sech} x$	$\rightarrow$	$\tanh x$	$\tanh x = \sqrt{1-\operatorname{sech}^2 x}$
$\csc hx$	$\rightarrow$	$\coth x$	$\coth x = \sqrt{1+\operatorname{csch}^2 x}$

• to pass semantically = algebraically in the trellis

from	to	replace	by
$\sin^{-1} x \rightarrow$	$\cos^{-1} x$	$x \rightarrow$	$\sqrt{1-x^2}$
$\cos^{-1} x \rightarrow$	$\sin^{-1} x$	$x \rightarrow$	$\sqrt{1-x^2}$
$\tan^{-1} x \rightarrow$	$\sec^{-1} x$	$x \rightarrow$	$\sqrt{x^2-1}$
$\cot^{-1} x \rightarrow$	$\csc^{-1} x$	$x \rightarrow$	$\sqrt{x^2-1}$
$\sec^{-1} x \rightarrow$	$\tan^{-1} x$	$x \rightarrow$	$\sqrt{1+x^2}$
$\csc^{-1} x \rightarrow$	$\cot^{-1} x$	$x \rightarrow$	$\sqrt{1+x^2}$

• to pass semantically = algebraically in the trellis

from	to	replace	by
$\sinh^{-1} x$ $\rightarrow$	$\cosh^{-1} x$	x $\rightarrow$	$\sqrt{x^2 - 1}$
$\cosh^{-1} x$ $\rightarrow$	$\sinh^{-1} x$	x $\rightarrow$	$\sqrt{x^2 + 1}$
$\tanh^{-1} x$ $\rightarrow$	$\mathrm{sech}^{-1} x$	x $\rightarrow$	$\sqrt{1 - x^2}$
$\coth^{-1} x$ $\rightarrow$	$\mathrm{csch}^{-1} x$	x $\rightarrow$	$\sqrt{1 + x^2}$
$\mathrm{sech}^{-1} x$ $\rightarrow$	$\tanh^{-1} x$	x $\rightarrow$	$\sqrt{1 - x^2}$
$\mathrm{csch}^{-1} x$ $\rightarrow$	$\coth^{-1} x$	x $\rightarrow$	$\sqrt{x^2 - 1}$

- to pass semantically = algebraically in the trellis

(1) between the exponential formulas
in 6 trig & in 6 hyp

and

(2) between the logarithmic formulas
in 6 trig^{-1} & in 6 hyp^{-1}

use the 48 trig - hyp conversion formulas
that are organized on nine sheets
and which may be displayed
in an aitch - shaped array

Δ biolines

• Euler, Leonard
1707 - 1783
Swiss (spent many years in Germany & Russia)
algebraist, analyst, geometer, number theorist,
probabilist, applied mathematician, calculating prodigy,
most prolific mathematician of all time
Euler = pr OI - ler

• Lambert, Johann Heinrich
1728 - 1777
Swiss - German
analyst, number theorist,
astronomer, philosopher, physicist;
gave the first systematic development of hyperbolic functions
and introduced their names & notation

Δ a note on the pronunciation
of unpronounceable functional abbreviations

• the following suggested spoken readings
of the functional notation seem to be more or less standard:

exp = ex - po	ln = nat log = log
e^x = e to the x	log = log

sin = sine = sign	sinh = sinch
cos = koss	cosh = kosch
tan = tan	tanh = tanch
cot = koh - tan	coth = koh - tanch
sec = seck	sech = setch
csc = koh - seck	csch = koh - setch

• also:

trig = trig
hyp = hype / hip
fcn = function

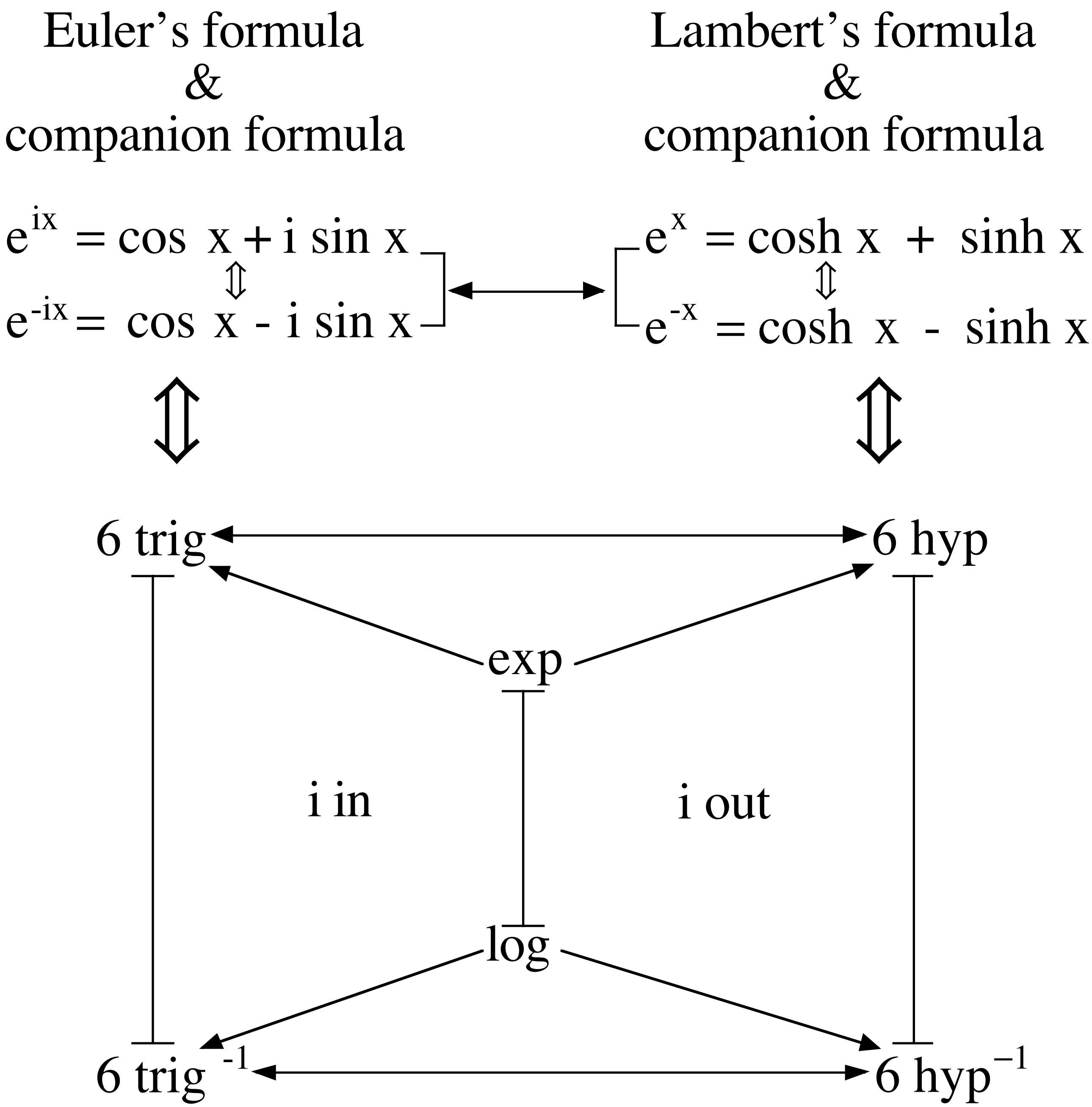
f^{-1} = inverse f where $f \in$ trig fcns $\cup$ hyp fcns

• an alternative would be
the full pronunciation of names for trig fcns
&
aitch / hype / hip sine etc for hyp fcns

Δ disclaimer

note that certain subtleties about
domains
ranges
square roots
multiple - valued functions
principal - valued inverse functions
logarithms of complex numbers
are deliberately omitted from consideration
for the sake of greater simplicity
in an initial overview

Einstein once remarked that
everything should be made as simple as possible but no simpler



the transcendental trellis

6 trig

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

$$\tan x = \frac{1}{i}\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = \frac{\sin x}{\cos x}$$

$$\cot x = i\,\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{2}{e^{ix} + e^{-ix}} = \frac{1}{\cos x}$$

$$\csc x = \frac{2i}{e^{ix} - e^{-ix}} = \frac{1}{\sin x}$$

6 trig

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

$$\tan x = \frac{1}{i}\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = \frac{\sin x}{\cos x}$$

$$\cot x = i\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{2}{e^{ix} + e^{-ix}} = \frac{1}{\cos x}$$

$$\csc x = \frac{2i}{e^{ix} - e^{-ix}} = \frac{1}{\sin x}$$

6 hyp

$$\sinh x = \frac{e^{x} - e^{-x}}{2} = \sinh x$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} = \frac{\cosh x}{\sinh x}$$

$$\mathrm{sec\,h}\, x = \frac{2}{e^{x} + e^{-x}} = \frac{1}{\cosh x}$$

$$\mathrm{csch}\, x = \frac{2}{e^{x} - e^{-x}} = \frac{1}{\sinh x}$$

6 hyp

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x}$$

$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}} = \frac{1}{\sinh x}$$

6 trig^{-1}

$$\sin^{-1} x = \frac{1}{i}\log\left(ix + \sqrt{1-x^2}\right)$$

$$\cos^{-1} x = \frac{1}{i}\log\left(x + i\sqrt{1-x^2}\right)$$

$$\tan^{-1} x = \frac{1}{2i}\log\frac{1+ix}{1-ix}$$

$$\cot^{-1} x = \frac{1}{2i}\log\frac{ix-1}{ix+1}$$

$$\sec^{-1} x = \frac{1}{i}\log\frac{1+i\sqrt{x^2-1}}{x}$$

$$\csc^{-1} x = \frac{1}{i}\log\frac{i+\sqrt{x^2-1}}{x}$$

6 trig^{-1}

$$\sin^{-1} x = \frac{1}{i}\log\left(ix + \sqrt{1-x^2}\right)$$

$$\cos^{-1} x = \frac{1}{i}\log\left(x + i\sqrt{1-x^2}\right)$$

$$\tan^{-1} x = \frac{1}{2i}\log\frac{1+ix}{1-ix}$$

$$\cot^{-1} x = \frac{1}{2i}\log\frac{ix-1}{ix+1}$$

$$\sec^{-1} x = \frac{1}{i}\log\frac{1+i\sqrt{x^2-1}}{x}$$

$$\csc^{-1} x = \frac{1}{i}\log\frac{i+\sqrt{x^2-1}}{x}$$

6 hyp^{-1}

$$\sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$$

$$\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}$$

$$\coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1}$$

$$\mathrm{sech}^{-1} x = \log\frac{1+\sqrt{1-x^2}}{x}$$

$$\mathrm{csc\,h}^{-1} x = \log\frac{1+\sqrt{1+x^2}}{x}$$

6 hyp^{-1}

$$\sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$$

$$\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right)$$

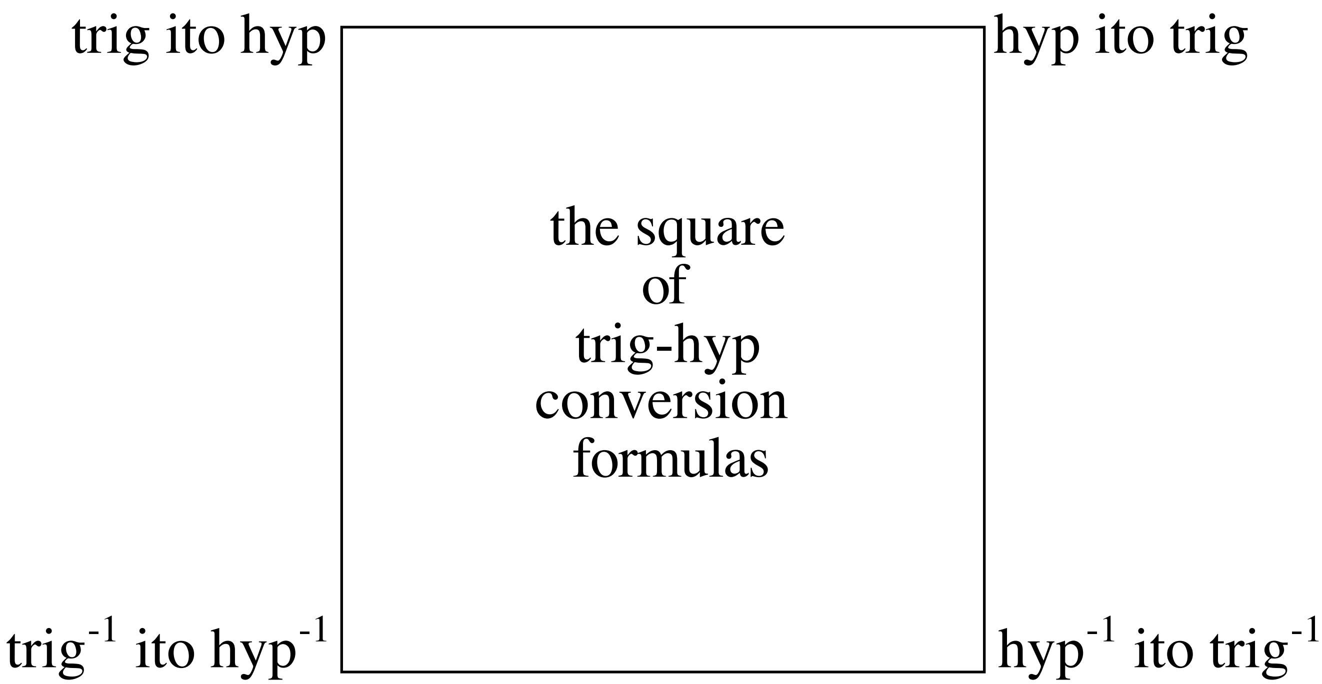
$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}$$

$$\coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1}$$

$$\text{sech}^{-1} x = \log\frac{1+\sqrt{1-x^2}}{x}$$

$$\text{csch}^{-1} x = \log\frac{1+\sqrt{1+x^2}}{x}$$

□ the trigonometric-hyperbolic conversion formulas



trig ix ito hyp x

$$\sin ix = i \sinh x$$

$$\cos ix = \cosh x$$

$$\tan ix = i \tanh x$$

$$\cot ix = \frac{1}{i} \coth x$$

$$\sec ix = \operatorname{sech} x$$

$$\csc ix = \frac{1}{i} \operatorname{csch} x$$

$\text{trig}\,x \ \text{ ito } \ \text{hyp}\,ix$

$$\sin x = \frac{1}{i}\sinh ix$$

$$\cos x = \cosh ix$$

$$\tan x = \frac{1}{i}\tanh ix$$

$$\cot x = i\coth ix$$

$$\sec x = \operatorname{sech} ix$$

$$\csc x = i\operatorname{csch} ix$$

hyp ix ito trig x

$$\sinh ix = i\sin x$$

$$\cosh ix = \cos x$$

$$\tanh ix = i\tan x$$

$$\coth ix = \frac{1}{i}\cot x$$

$$\operatorname{sech} ix = \sec x$$

$$\operatorname{csch} ix = \frac{1}{i}\csc x$$

hyp x ito trig i x

$$\sinh x = \frac{1}{i} \sin i x$$

$$\cosh x = \cos i x$$

$$\tanh x = \frac{1}{i} \tan i x$$

$$\coth x = i \cot i x$$

$$\operatorname{sech} x = \sec i x$$

$$\operatorname{csch} x = i \csc i x$$

$\operatorname{trig}^{-1} ix$ ito hyp^{-1}

$$\sin^{-1} ix = i \sinh^{-1} x$$

$$\cos^{-1} ix = i \cosh^{-1} ix$$

$$\tan^{-1} ix = i \tanh^{-1} x$$

$$\cot^{-1} ix = \frac{1}{i} \coth^{-1} x$$

$$\sec^{-1} ix = i \operatorname{sech}^{-1} ix$$

$$\csc^{-1} ix = \frac{1}{i} \operatorname{csch}^{-1} x$$

$\text{trig}^{-1}x$ ito hyp^{-1}

$$\sin^{-1}x = \frac{1}{i}\sinh^{-1}ix$$

$$\cos^{-1}x = i\cosh^{-1}x$$

$$\tan^{-1}x = \frac{1}{i}\tanh^{-1}ix$$

$$\cot^{-1}x = i\coth^{-1}ix$$

$$\sec^{-1}x = i\,\text{sech}^{-1}x$$

$$\csc^{-1}x = i\,\text{csch}^{-1}ix$$

$\text{hyp}^{-1}\, \text{i}x \ \text{ ito } \ \text{trig}^{-1}$

$$\sinh^{-1} \text{i}x = \text{i}\sin^{-1} x$$

$$\cosh^{-1} \text{i}x = \frac{1}{\text{i}}\cos^{-1} \text{i}x$$

$$\tanh^{-1} \text{i}x = \text{i}\tan^{-1} x$$

$$\coth^{-1} \text{i}x = \frac{1}{\text{i}}\cot^{-1} x$$

$$\text{sech}^{-1} \text{i}x = \frac{1}{\text{i}}\sec^{-1} \text{i}x$$

$$\text{csch}^{-1} \text{i}x = \frac{1}{\text{i}}\csc^{-1} x$$

$\text{hyp}^{-1} x$ ito trig^{-1}

$$\sinh^{-1} x = \frac{1}{i} \sin^{-1} i x$$

$$\cosh^{-1} x = \frac{1}{i} \cos^{-1} x$$

$$\tanh^{-1} x = \frac{1}{i} \tan^{-1} i x$$

$$\coth^{-1} x = i \cot^{-1} i x$$

$$\text{sech}^{-1} x = \frac{1}{i} \sec^{-1} x$$

$$\text{csch}^{-1} x = i \csc^{-1} i x$$

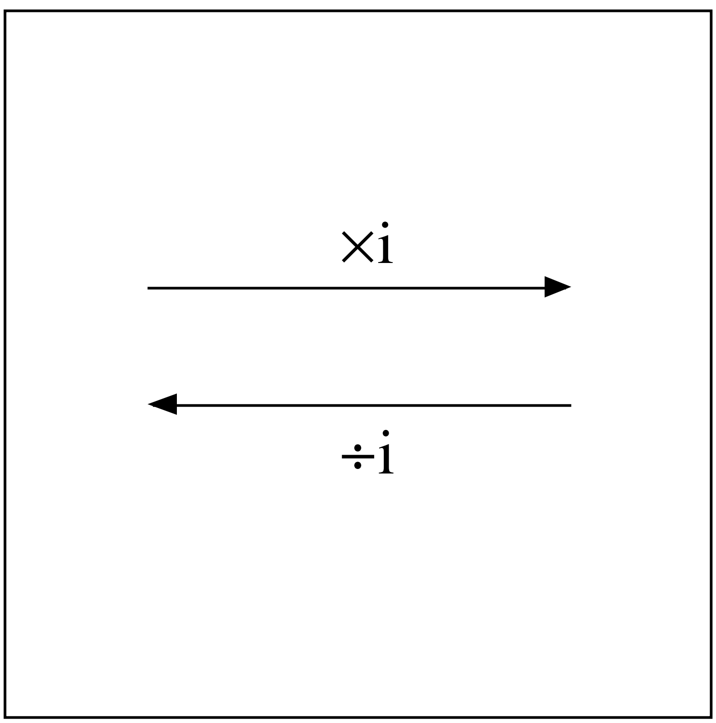
☐ the squares of trig-hyp special values may be constructed by making use of the preceding formulas

eg

$$\sin x = \frac{1}{i} \sinh ix$$

$$i \sin x = \sinh ix$$

$\sin \frac{\pi}{2} = 1$ $\qquad$ $\sinh i \frac{\pi}{2} = i$



$\sin^{-1} 1 = (2Z + \frac{1}{2})\, \pi$ $\qquad$ $\sinh^{-1} i = i\, (2Z + \frac{1}{2})\, \pi$

$$\sin^{-1} x = \frac{1}{i} \sinh^{-1} ix$$

$$i \sin^{-1} x = \sinh^{-1} ix$$